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ABSTRACT

The aim of this monograph is to describe a program of research investigating the problem-solving responses of young children and to present in summary form the results of the analyses completed so far. Attention is given to characteristics of clinical studies which are thought to make the clinical method particularly useful with young children. After a brief introduction, the first section describes the research decisions that had to be made either prior to launching the study or while it was being conducted. The second section summarizes results obtained in unique problem situations involving division, space and geometry, reflections in a plane, mirror reflection, coordinates, and factors and sequences. Problem situations devised for the studies, such as cargo groups, fold-out shapes, parking grids and theater grids, are described and illustrated in color photographs and diagrams. Additionally, for most problem situations the mathematical significance of the problem is pointed out and results and conclusions or concluding statements are given. The final section of the monograph evaluates the clinical methodology employed in the studies and considers the contribution of the research to mathematics education. (Author/RH)

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STUDYING PROBLEM SOLVING BEHAVIOR IN EARLY CHILDHOOD

Doyal Nelson

Lecture Series Award Monograph

The University of Alberta
Faculty of Education
Edmonton, Alberta
Canada T6G 2G5

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FOREWORD

Dr. L. Doyal Nelson is the first recipient of the Faculty of Education Lecture Series Award. This award was established to promote and publicize educational research by the Faculty. It is intended to reward those staff members who have gained a wide reputation for contributions to knowledge in their field through a sustained program of research, and who have consistently disseminated the results of their research in publications and oral presentations. In addition, the recipient must have identified a significant related research project that can be undertaken and reported as a result of receiving the award. The award carries with it a six-month leave to work on the project with the expectation that it will be culminated by a public presentation and a publication.

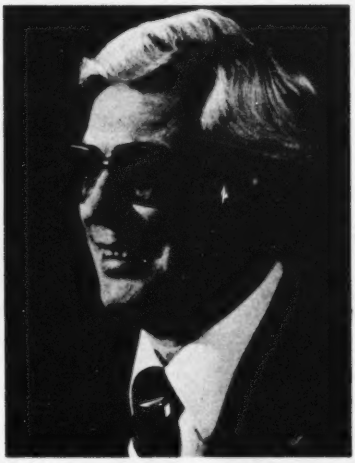
In January, 1980 Dr. Nelson delivered two public lectures on his work. One was entitled "*Diagnosing Learning Difficulties in Elementary School Mathematics*"; the other dealt with "*Studying Problem-Solving Behavior in Young Children*". These lectures were based upon investigations extending over a decade, and reported on in this monograph.

Readers of the monograph will encounter valuable insights and speculations of relevance for classroom practice about the problem-solving strategies of young children. They will also encounter a promising methodology for research in problem-solving that warrants further application and refinement. But above all, they will be stimulated to repeatedly ask the most simple yet the most profound question that can advance our knowledge about problem-solving behavior: Why?

We are pleased and proud that Doyal Nelson has launched our Lecture Series Award in such an auspicious manner.

W.H. Worth
Dean, Faculty of Education
University of Alberta

April, 1980



VITAE

Dr. Leonard Doyal Nelson

A Bibliographic Sketch

Doyal Nelson was born in Lethbridge, Alberta on July 25, 1920. He received his elementary and secondary education in Wrentham School in Southern Alberta. Doyal embarked on a life's work in teaching by attending the Calgary Normal School in 1939-40. Following a year of teaching in a one-room rural school, he enlisted in the Royal Canadian Air Force. On his return from overseas in 1945, he enrolled in the Faculty of Education, University of Alberta, Edmonton, and obtained the B.Ed. and M.Ed. degrees from this institution.

In 1949, Doyal returned to Southern Alberta as a secondary school teacher. In 1950 he became principal of the Nobleford School. Following three years of service in that capacity, he joined the staff of the provincial Department of Education as a superintendent of schools. He was assigned to the County of Forty Mile with an office in Foremost, Alberta.

Doyal began the second phase of his career in 1959. At that time he joined the staff of the Department of Elementary Education at the University of Alberta. In 1962 he obtained the Ph.D. from the University of Minnesota. Dr. Nelson has served and represented the University of Alberta in many ways and in many capacities. He was acting chairman of the Department of Elementary Education in 1968-69. He served on General Faculties Council and on the University Senate as well as on many Faculty and University committees.

In pursuing his research interests in mathematics education, he has been recognized in the following ways:

1971 - he became an Associate with the National Institute of Educational Research, Tokyo; 1973 - a U.N.E.S.C.O. consultant in Bangkok; 1974 - a participant in the Conference of the International Congress of Mathematics Instruction, Tokyo; 1976 - Chairman of the Research Section, 3rd International Congress of Mathematics Instruction, Karlsruhe, Federal Republic of Germany. More recently he has served as a member of the Board of Directors of the National Council of Teachers of Mathematics and as Chairman, Research Section, Science Council of Canada Conference on Mathematics Instruction. Dr. Nelson has also held short term appointments at McGill University, the University of Minnesota, and at California State University, Northridge.

Throughout Dr. Nelson's distinguished career as a teacher and as a teacher educator, he has held an abiding interest in the nature and development of the behavior of younger children as they tackle problems in mathematics. The research reported in this monograph earned the support of Canada Council. Doyal is continuing his work in this field at the University of Alberta.

TABLE OF CONTENTS

INTRODUCTION xi

I. RESEARCH PROCEDURES 1

A. Overview 1
B. Formulating the Research Question 3
C. Design and Implementation 5
D. Analysis and Interpretation 10

II. SUMMARY OF RESULTS 12

A. *Division* 12

i. Cargo Groups 12
ii. Animal Groups 13
iii. Mathematical Significance 14
iv. Discussion of Results 15
v. Some Concluding Statements 19

B. *Space and Geometry* 20

i. Fold-out Shapes 20
ii. Projected Shapes 22
iii. Mathematical Significance 24
iv. Discussion of Results 24
v. Some Concluding Statements 29

C. *Reflections in a Plane* 30

i. Object Reflection 30
ii. Mirror Reflection 30
iii. Mathematical Significance 31
iv. Discussion of Results 31
v. Some Concluding Statements 33

D. *Co-ordinates* 33

i. Parking Lot Grid 34
ii. Theatre Grid 34
iii. Mathematical Significance 35
iv. Discussion of Results 36
v. Some Concluding Statements 37

E. *Factors* 38

i. Factor Platform 38
ii. Factor Board 39
iii. Mathematical Significance 41
iv. Discussion of Results 41
v. Some Concluding Statements 44

F. *Sequences* 45

i. Circular Sequences 45
ii. Linear Sequences 47
iii. Mathematical Significance 48
iv. Discussion of Results 49

III. EVALUATION 51

A. Methodology 51
B. Contributions of Study to Problem Solving Research 54
C. Concluding Statement 58

BIBLIOGRAPHY 59

INTRODUCTION

In the past decade and a half the work of Jean Piaget has had a profound effect upon the course of research in the teaching and learning of elementary school mathematics. From the early sixties to the early seventies most of the formal research in mathematics education conducted in the Department of Elementary Education at the University of Alberta dealt with some aspect of Piaget's work. (Brace, 1963; Cathcart, 1969; Leidtke, 1968; Newman, 1967; Pelletier, 1966; Reimer, 1968; Sawada, 1966; Towler, 1965, 1967;). In addition to the formal work there were many hours spent informally with children individually, in small groups and in classroom settings in the attempt to see the relevance of Piaget's work to elementary school mathematics.

The majority of studies have been concerned with Piaget's concept of conservation and its role in learning mathematics. The notion of conservation of number (discrete quantities) was described by Piaget (1952) and was further elaborated to include the notion of conservation of continuous quantities in a later work on geometry (Piaget and Inhelder, 1960). Piaget's contention that conservation is the basis of rational activity along with his ingenious methods of working with children opened up what appeared to be a fertile area of research for mathematics educators.

At first there was skepticism about the appropriateness of Piaget's methods, the accuracy of his observations and his interpretations of children's responses. As a result, many of the early studies in mathematics education here were either replications of his early work or derivations from it. As the notion of conservation became more clearly understood, efforts were made to determine how conservation affected the learning of mathematics in young children. A review of the formal work in mathematics education in the Department of Elementary Education over most of the period has been prepared by Cathcart and Liedtke (1969) and need not be repeated here.

Ten years of research have failed to establish any practical connection between conservation and mathematics learning. What is an essential and central concept in Piaget's developmental psychology seems to have little bearing on how children learn mathematics. Although it may appear unwise to have followed such a tenuous lead for ten years, the time was not at all wasted. During that time considerable skill was gained in using clinical methods similar to those employed by Piaget. We also learned a good deal about how to construct problem situations, to which children could respond. Through analysis of such responses it became

increasingly apparent that, in the early stages at least, mathematical knowledge is constructed by the individual through interaction with real objects in practical or real situations.

In the early seventies much of our effort went into devising and refining criteria or guidelines for the creation of "good" problem situations for children in the age range 3 years to 8 or 9 years. As the guidelines developed, protocols for mathematical tasks associated with the problem situations were worked out with individual children over the age range.

In March 1973 a formal proposal was submitted to Canada Council for support for a project entitled, "The Nature and Development of Problem Solving Behavior in Early Childhood". The first phases of the project were funded and supplementary funds were granted which permitted analysis of responses to most of the problems included in the proposal.

The object of this monograph is to describe the project whose title is given above and to present in summary form the results of the analyses completed so far. Some space will also be devoted to discussing characteristics of clinical studies which make the methodology particularly useful for use with young children.

The monograph is intended for beginning research workers interested in using clinical procedures for studying the cognitive functioning of children. Practicing elementary school teachers attempting to develop problem solving skills in young children should find the suggestions for creating good problems and the actual problem examples useful in their work.

The next section contains a description of the research decisions that had to be made either prior to launching the study or while it was being conducted.

RESEARCH PROCEDURES

A. OVERVIEW

The problem solving behavior of children aged three to nine years was the domain of the study to be discussed here. Within the domain the following research tasks served as areas of focus. First, it was necessary to identify, select and refine criteria for specifying and constructing mathematical problems which could be used effectively to study the problem solving behavior of such children. Once the criteria had been established problem situations were selected and constructed. Then children were engaged, under carefully controlled clinical conditions, in interacting with the problem situations. Behaviors manifested by the children as they were thus engaged were observed, recorded, coded and categorized. Finally the development of problem solving behavior over the age range was traced.

In order to make the research tasks more specific, a number of constraints were imposed upon the domain. The first constraint was on the content. Each problem situation had a definite mathematical interpretation or was the embodiment of some basic mathematical idea. In that sense each was content specific. The second constraint concerned the mode of interaction. Most other research in mathematical problem solving had occurred at a highly verbal level; the problem situations being presented verbally and the solution expressed symbolically. In this research it was required that solutions always be expressible in terms of physical manipulation or movement. Thus, each problem situation needed to be a concrete embodiment of some mathematical idea and the solution amenable to expression through the manipulation of various components of the embodiment. Such manipulative behaviors were the prime focus. It was not the purpose to explain thinking or necessarily to explain the processes involved in problem solving. This level of interpretation placed a third constraint on the research. The modest aim forced complete attention on the spontaneous manipulative and verbal responses of the children as they solved the problems. A final constraint was in the age range of children to be studied. In Piagetian terms children in the range three to nine years would be either in the pre-operational or concrete operational stage of development. If effective communication were to be established and maintained over the age range the concrete embodiment constraint mentioned above was essential.

The establishment of a set of criteria on which to base the development of problem situations to meet the requirements was accomplished over a period of years and a workable set was reported in the Thirty-seventh Yearbook of the National Council of Teachers of Mathematics (Nelson and Kirkpatrick, 1975 and Nelson and Sawada, 1975). The set of criteria is given below.

1. *The problem should be of significance mathematically.* It is the potential of the situation as a vehicle for the development of mathematical ideas that we choose a particular problem situation or family of situations above all others.

2. *The situation in which the problem occurs should involve real objects or obvious simulations of real objects.* The main consideration here is that it be comprehensible to children and easily related to their world of reality.

3. *The problem situation should capture the interest of the child either because of the nature of the materials, the situation itself, the transformations children can impose on the materials, or because of some combination of these factors.* Observations will readily establish whether this criterion is met or not. It is not always clear from observation, however, which factor or combination of factors is generating interest in the child.

4. *The problem should require children to make moves or transformations or modifications with or in the materials.* It is difficult to overemphasize the role of action in early childhood learning. Most of the mathematical models we are interested in at this level are what might be called "action models."

5. *Wherever possible, problems should be chosen which offer opportunities for different levels of solution.* If children can move immediately from the problem situation to an expression of its mathematical structure it is not a problem.

6. *Whatever situation is chosen as the particular vehicle for the problems, it should be possible to create other situations which have the same mathematical structure.* Another way of saying this is that the same problem should have many physical embodiments. It may not be possible for children to generalize a solution to a certain kind of problem until the problem has come up in a variety of situations.

7. *Finally, children should be convinced they can solve the problem and should know when they have a solution for it.* If children are somehow required to react with or transform materials used in problem situations it is usually easy to determine whether the problem meets the criterion or not.

Using the seven criteria as the basis, a group of six problem solving situations involving different mathematical ideas were selected from a number which had been designed. For each of these six problem situations a second concrete and equivalent embodiment was constructed. The basic problem situations, but not their equivalents, are described by Nelson and Sawada (1975).

These 12 problem situations were presented to 15 children at each age level, three to eight, according to a pre-arranged schedule. Each child in this sample of 90 children was presented wth six problem situations; four of these were different and two were the equivalents of two of the first four. A year later, as many children as were available were brought back and asked to do six more problems. There were 74 children in this second group. The particular problem situations which the children were presented with in both the cross-sectional and the longitudinal sampling were established by the sampling schedule mentioned above. In both the cross-sectional and longitudinal sampling, children were seated, one at a time, at a low table with an interviewer to their right. The interviewer presented the problems one after another at one sitting which lasted from 20 minutes to a little over an hour. Children's manipulative and verbal behaviors were recorded on one-half inch video tape. Two black and white cameras and a device to provide a split image were used in order that two views of children, if desired, could be recorded at once. All cross-sectional data were recorded on 54 one-hour tapes; the longitudinal data on 45 one-hour tapes.

In the cross-sectional sampling each problem situation was presented to 10 children at each age level. Five children at each level did the equivalent problems. The taped section for each child engaged in a particular problem was separated out and analysed. There were 60 taped sections for each problem and 30 for each equivalent for the cross-sectional aspect. These 60 taped sections were analysed as a unit. Complete reports of such analyses are available for 8 of the 12 problems (Little, 1976; Bourgeois, 1976; Fenrich, 1978). Summaries are also reported for four of the problems in Nelson and Kieren (1977) and Bourgeois and Nelson (1977). Other analyses are in various stages of preparation.

No pre-conceived system for coding the taped data to written form was established. It was required that the form of the code would be determined solely by viewing the tapes. A unique code had to be developed for each separate problem and samples of the codes are provided by Bourgeois (1976), Little (1976) and Fenrich (1978). Transferring behavioral data from tape to written form proved to be an arduous task.

Once the data were coded, behaviors were described and categorized using only the information from the tapes. A sample of the findings from such an analysis can be found in the papers by Bourgeois and Nelson (1977) and Nelson and Kieren (1977).

B. FORMULATING THE RESEARCH QUESTION

Unlike most reported research there was no attempt to formulate hypotheses which could then be tested through familiar statistical procedures. The primary aim was to determine how young children (ages three to nine) would behave or respond when faced with specific mathematical problems centering around or embodied in specific apparatus.

The physical situations were constructed in such a way that it was always possible to do the related problem tasks by manipulating the elements of the situation. This did not, of course, prevent any child from giving a more sophisticated or, at least, a more symbolic solution. Indeed, the problems were in the main so constructed that a range of solutions from highly manipulative to purely symbolic was possible. As subsequent data were gathered, it became increasingly apparent that the presence of the apparatus and the nature of the questions directed to the child tended to encourage a manipulative rather than a symbolic response.

To err in the direction of manipulation was considered to be more desirable than to err in the other. Previous work had suggested that even three-year-olds had considerable control over many mathematical processes which they could express through their manipulations but not at all in any symbolic form. Decreasing the probability of a manipulative response rather than a verbal or more symbolic one increased the probability that the younger children would respond if they had any knowledge or understanding of the underlying mathematical process or idea. It was assumed that older children who could respond in a more symbolic form would do so. Actual recorded responses of the children cast considerable doubt on the validity of this assumption. Older children who were capable of making a symbolic response as often as not gave a manipulative response and would produce a symbolic one only when urged to do so or they would give both.

One of the most difficult aspects at this stage was to formulate the set of criteria which would serve as guidelines for the construction of the problem situations and for the formulation of the associated problem tasks. The set of 7 criteria outlined in the overview were developed out of a study of the literature and many hours of careful clinical work with children. Still, at this stage, the criteria are very crude and though they do provide guidance for constructing physical apparatus and associated problem tasks they need further refinement. It is clear that some problems constructed are "better" than others in that some conform more closely to the guidelines than others. Unfortunately the problems that seem to be better when measured against the guidelines do not produce problem solving behaviors on the part of children distinguishable in quality from problems which do not so measure up.

This is certainly not the first attempt anyone has ever made to define what a problem is or to provide criteria for producing suitable problems. These attempts for the most part have had only limited success and application. The attempt here is no exception. However, crude as these guidelines are, they do help in formulating the associated problem tasks.

With the guidelines thus established it was possible to ask an appropriate research question. In this case it was:

What is the nature and development of problem solving behavior of young children while solving problems constructed in accordance with the guidelines?

I hasten to point out that the reader should not get the impression that the important research question arose after the guidelines were developed. In fact the question had occurred in a number of different forms long before, but always in connection with specific clinical encounters with children. These encounters gave every hope that one could tap the considerable mathematical understanding and problem solving ability of very young children if one only had a better method. Previous attempts had failed because the means adopted always required the child, to a greater or lesser extent, to produce some kind of written or symbolic response. The symbol system of young children is just not well enough established or extensive enough to permit them to make adequate responses in this form. Analyses of any such responses could only lead to grave distortions. Development of the guidelines was an attempt to express more precisely what kind of situations and what form of responses would be more appropriate for very young children. Any reservations expressed here about the application of the guidelines should be taken to indicate that there is still more work to be done in this area.

C. DESIGN AND IMPLEMENTATION

Informal clinical work with children over several years had established the basic form of some of the problems used. For the most part though, the problems were designed in their original form specifically to provide information on the research question posed in the previous section.

The problem which was called Cargo Groups (Bourgeois and Nelson, 1977) is an example of one which was developed and refined over a period of years. In one of its early versions it consisted of a ticket board base with a blue construction paper cutout running across to represent a river. The ferry boat was a ticket board cutout and the "cargo" was made up of a number of cutout blocks two centimetres on a side. The protocols used to present the situation to the child and to describe the problem tasks also had undergone a number of refinements. The equivalent situation which came to be known as Animal Groups (Bourgeois and Nelson, 1977) was designed specifically for this study.

The method used in creating the majority of the problems can best be described by following through the process for a specific problem. The example used will be what came to be known as Fold-Out Shapes. For a description of the

apparatus see Nelson and Kieren (1977). To specify the mathematical idea or process involved for all of the problems a team made up of the research director, an associate and four graduate students shut themselves in a classroom for a pre-determined period usually on the order of two or three hours for three or four days in a week.

Mathematical content considered appropriate for young children was outlined on a blackboard and appropriate related problem situations suggested for each. Application of the guidelines in developing problem situations and tasks was a fundamental requirement. Outlining content proved not too controversial or difficult a task but it took some time for all members of the team to learn how to apply the criteria specified in the guidelines. Since the Cargo Groups apparatus was already available it served as a good model for the application of guidelines and was referred to often in the early stages.

Several geometry topics were outlined under mathematical content and one concerned the construction of three-dimensional solids from two-dimensional elements. In order for children to see the relationship of the elements to each other, and how they came together to form a network which could then be folded to make a three-dimensional shape, it was necessary that the elements be easily attached to each other and as easily taken apart. With the cube, for example, there were six squares and it should be easy for children to see them as individual pieces but it should also be easy for them to assemble them together to form the cube. At the same time, it should be just as easy for them to go the other way. There are several commercial sets which have elements that can be put together to form three dimensional shapes but one design problem is that they cannot be easily assembled and disassembled.

There is a Technical Services branch at the University of Alberta which normally deals with design problems such as this so the first step was to approach them. Their initial suggestion was to use Plexiglas elements and imbed magnets at intervals along the sides. This would certainly meet the requirements of easy assembly and disassembly but the resulting apparatus would likely be too bulky for a three-year-old to handle.

To try to resolve this difficulty, the group went back again to the closed classroom and, as it happened, in one session resolved it. In discussing possible means of joining elements someone mentioned the material Velcro. It is a cloth zipper whose most familiar use in this part of the country is to close pockets on winter clothes. One side of the zipper consists of a mass of small loops woven into tape, the other consists of a mass of tiny hooks also woven into tape. When the two sides are brought into contact, the hooks engage the loops and make a positive join. A slight tug will separate the two sides. The ideal material for our purpose!

A four-foot by eight-foot sheet of one-eighth inch Plexiglas was purchased along with several widths and colors of Velcro. Six ten-centimetre squares were cut from the Plexiglas and alternate sides of Velcro contact cemented to the edges of the squares. A similar procedure was used for triangles and pentagons using the constant ten-centimetre dimension. (The reader is left with the problem of how to attach Velcro to elements with an odd number of sides.) With six squares, 20 triangles and 12 pentagons it is possible to construct all the Platonic solids and by combining the various elements in different ways a number of other solids as well.

There are a few additional design problems which had to be solved even after the prototypes had been constructed. A saw was used first to cut the Plexiglas but a cutter can be obtained which scribes the plastic as a glass cutter scribes glass. The plastic breaks into straight neat lines as glass would. Using ordinary contact cement was found not to be suitable for attaching the Velcro to the plastic. After trying a number of glues and cements without success it was found that there is a premium grade contact cement available and that Velcro glued to Plexiglas with that cement is there to stay. Various widths of Velcro were tried but it was found that three-quarter inch Velcro on the one-eighth inch Plexiglas served best.

The reader may think that design problems such as the ones described are not the province of research workers in mathematics education and should be referred to design experts for solution. There are no doubt very creative people available to do this if they can be found. Finding them is certainly not easy. In addition, research workers must be prepared to pay high rates for such services and budgets are, more often than not, too tight to provide the necessary funds. Conveying the notion of precisely what is needed to even highly qualified designers is often difficult and creates further difficulties. We were fortunate enough to find a carpenter who could translate our specifications, descriptions, and crude drawings into highly attractive and creative apparatus. His skills were precisely what we needed.

Written protocols for presenting apparatus and for outlining the associated problem tasks were developed through extensive work with young children in the laboratory and in schools. In the early stages it was estimated that children might be able to deal with about eight different problem situations in one sitting. It turned out, however, that the session for some became over-long and the number of problems per sitting was reduced to six. In developing the protocols and later in the actual research situation the child would sit at a low table in the laboratory beside an interviewer and the problems would be presented one after the other.

The laboratory set-up and procedures were refined through a large number of pilot runs. Clinical procedures proved interesting to children and to their parents with the result that at this stage there were more volunteers than could be used. At the outset a decision was made to record children's responses on video tape and to

use these tapes as the sole basis for subsequent analysis. A number of people on the research team had had considerable experience with recording behaviour on tape and film and gave indispensable advice at this stage. It was apparent that color in some cases influenced the way children responded but since color facilities were not available a compromise had to be made. Children's responses were recorded on black and white video tape and key sequences filmed on 8mm color film. We were never able to overcome the difficulty of co-ordinating the taped and filmed sequences so had to abandon the use of color film entirely.

With the assistance of competent technical personnel two video cameras were trained on the table at which children worked, one giving a close-up view of the action and the other a wider angle view. There was a split-image capability so that images from both cameras would be recorded at once. Technicians were trained to operate the cameras and to catalogue the recorded data. Two members of the research team did all the interviewing.

Aside from inevitable technical breakdowns (which in our case were few indeed) there were no difficulties at all with taping. Every part of the procedure had been carefully systematized through hours of work with children before the actual research data were collected. It was also a considerable and time-consuming task to systematize the protocols used to present the problems and problem tasks to children. Unfortunately in the cross-sectional data a rather serious procedural error was introduced. The reader will probably have already noted that there was a fear on our part that some informaton would be lost through children not responding either through unfamiliarity with the apparatus, uneasiness with the situation, or for some other reason. It was decided that when a child would not respond, the interviewers at their discretion, could make minor changes in the protocol. Early in the collection of the cross-sectional data it became clear that our fears were unfounded. Very few children failed to respond but deviations from protocol occurred for those who did not at first respond. We did not want to tighten up the protocols at this stage and didn't have the resources or time to start over. Subsequent analysis of the data proved to be difficult wherever there were deviations from the established protocols and the only way to handle this was to describe in detail precisely what deviations had occurred. When the longitudinal data were collected, the established protocols were followed faithfully, making analyses much simpler. Researchers who attempt clinical studies are strongly advised to develop protocols that vary from subject to subject as little as possible or not at all; or, alternately, to structure interview protocols for each individual and report each as a case study.

Decisions now had to be made about the sample. As mentioned before, it was found that children could do six problems. The number of children at each level had to be between 10 and 20 to stay within the budget and at the same time to provide enough data for the analysis. The decision was made that each child

should do four different problems and two equivalents. Since four different problems can be chosen in ${}^{6}C_{4}$ or 15 different ways it was advantageous to require 15 subjects at each age level for a total of 90 subjects. This group provided the cross-sectional data. Then a year later, as many subjects as possible would be brought back. At this time they would do four equivalents and two original problems according to the same sampling scheme. The reader will be able to write out the 15 different patterns or problem sets for the cross-sectional data and the 15 for the longitudinal data. Each one of these problem sets would be randomly assigned to the subjects as soon as they were chosen. (For elaboraton of sampling procedure see Little, (1976)) It is clear that the scheme would permit a number of checks. The growth of children could be traced on a particular problem over the year. The effect of order of presentation could be checked because in the cross-sectional sampling the problem was given before its equivalent while in the longitudinal sampling the equivalent problem was presented first then the original. The longitudinal data would serve as a reliability check on the cross-sectonal data. For example, whatever could be said about how the four-year-olds differed as a group from the five-year-olds when looking at the cross-sectional data could be checked by looking at the difference between the four-year-olds in the cross-sectional sampling and the same children when they were five a year later. The check could not, of course, be perfect because in longitudinal sampling not all of the original children were available. In this study there were, in fact, 16 children lost between the first and second sampling.

To provide data for the purpose described it was assumed that children could be chosen from a volunteer group. While the characteristics of such a group were not expected to be completely representative, such a shortcoming was assumed to have minimal effect. Then, too, the difficulty of getting a random sample of three- and four-year-olds in any community is well known. All findings in the study would have to be checked further by more rigorous means in any case.

In May and June of 1974, 600 notices were distributed to parents of children in grades one, two and three in five elementary schools and two day-care centres in the vicinity of the University of Alberta. These notices invited parents to permit their children, ages three to eight years, to participate in the data collection in the summer of 1974 and 1975. Except for the three-year-olds, sufficient numbers of children were volunteered by their parents to allow selection by a random process. There were not enough three-year-olds to make up the sample of 15 so a number of personal requests were made to get the required number. The allocation of sets of problems to children within each age group was randomized. These sets were numbered 1 to 15 according to their occurrence in the systematic table used to construct the sets. Parents of the children selected were contacted by telephone in the order in which their applications were received and an appointment schedule was compiled. The first appointment for the cross-sectional sampling was for June 24, 1974, and the final interview was held on July 23, 1974. Similar procedures

were used for the longitudinal sampling. The interviews for this phase were conducted from July 3 to July 25, 1975. An attempt was made to keep to a schedule of three interviews in the morning and two in the afternoon. No three-year-olds were interviewed in the afternoon. Seven adults were normally in the room during the interview: two cameramen, the interviewer, two assistants, the director or associate director of the project and a parent of the child. Once they became involved in the problems no subject appeared to be unduly distracted by the rather formidable laboratory set-up.

D. ANALYSIS AND INTERPRETATION

For the cross-sectional and longitudinal sampling there was a total of nearly 100 hours of taped material. For analysis, each individual problem was separated out. In the cross-sectional sample there were 10 children at each level who did each of the original problems and five children who did the equivalents. If all children had been available for longitudinal sampling, 10 children would have done the problem equivalents and five would have done the original problems. The attrition rate has already been mentioned.

Separating out the taped sections for each problem involved a great deal of tape changing in addition to the viewing time required to get the children's responses on paper in some kind of manageable form. Each problem required its own coding system. After viewing a number of taped sections for children at different levels, categories of solution strategies began to emerge. These categories formed the basis of the analyses but individual behaviours had to be detailed within the categories. One of the most serious problems at this stage was to find coders who were able to go through this tedious process of analysis and still remain alert for significant periods of time. Tolerance levels ranged from only a few minutes to as much as three or four hours. The analyses for some of the problems have been completed but for others they are still in progress. Funding authorities and referees have to be convinced of the cost of such open-ended analyses.

At this juncture a word should be said about the risks involved when no *a priori* scheme is adopted for analysing data. In the first place, there is no guarantee that after all the effort put into collecting the data (in this case video tapes of children's responses) there will emerge any sensible and interpretable patterns. In fact, this led to problems with referees who could not accommodate themselves to such tentativeness. This in turn affected the funding agency which, on the advice of one referee, made a drastic and crippling cut in the requested budget. Fortunately the decision to analyse problem by problem enabled completion

of important parts of the study before the funds ran out. When there are departures from accepted methodologies - as certainly was the case here - it would be a rare occurrence for the agency to assign a completely sympathetic referee.

Examples of data analyses and interpretations are given in Bourgeois and Nelson (1977) and in Nelson and Kieren (1977).

SUMMARY OF RESULTS

A. DIVISION

The two problem situations, involving division and constructed to be mathematically equivalent became known as Cargo Groups and Animal Groups.

i. Cargo Groups

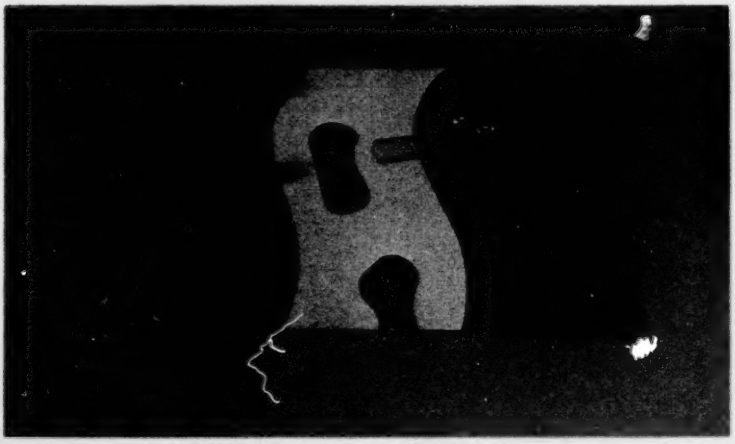


Illustration 1

The apparatus consisted of a masonite base 76 cm x 106 cm x 6 mm. At each side of the board were glued 4 mm masonite sections leaving a sinuous strip (to represent a river) 35 mm wide across the middle of the board.

Two irregular shapes to represent islands were made of 3 mm masonite and glued to the middle area. The river was light blue and the islands green. Large parking lots on each side of the river were painted in black and the rest of the board was painted green.

A simulated ferry boat (11 cm x 6.5 cm x 3 mm) with one concave end and one convex end was made of masonite. There were also 15 plastic cars of various colours and makes all about 5.3 cm long and 2 cm wide. Three wooden blocks; one blue, one red and one brown 5 cm x 5 cm x 2.5 cm; to represent houses completed the apparatus.

The cars were placed randomly in the parking lot closer to the subject. The child was asked how many trips would be needed to get the cars across the river to the other parking lot if the ferry could take them three at a time. After that task was completed the child was asked how many cars would be at each house if there were the same number of cars at each house.

ii. Animal Groups

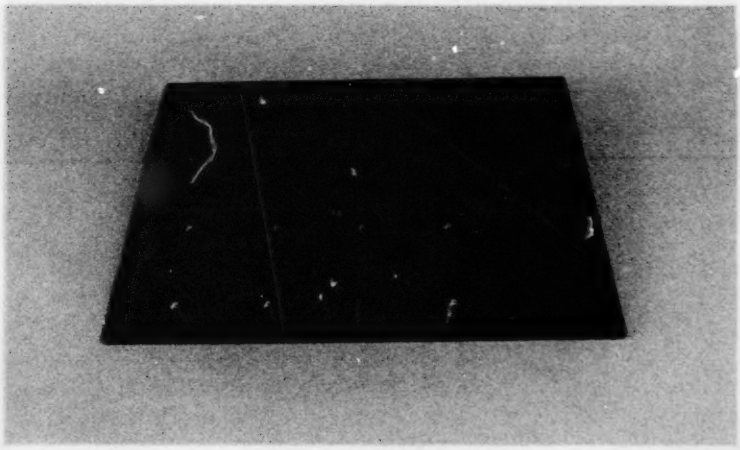


Illustration 2

The apparatus consisted of a base board, posts, walls and plastic animals.

The baseboard consisted of 2 cm plywood 76 cm x 107 cm with 25 holes of 18 cm centres drilled in a square array.

Dowels to form posts had four slots cut symetrically around the circumference while the diameter of one end was reduced to fit into the holes drilled into the board. Walls were made of 16.5 cm x 7.5 cm x 6 cm masonite slabs which would easily fit into the post slots. Twenty plastic animals from 4 to 5 cm tall completed the problem apparatus.

The set of 20 toy animals was presented to the child who was asked to build enough cages so there would be 5 animals in each cage and so there would be no animals left over. After the task was completed some cages were dismantled or some constructed so that there were 3 cages. Two of the animals were removed leaving 18 animals. The child was then asked how many animals would be in each cage if there were the same number in each.

iii. Mathematical Significance

The reader will recognize the first tasks in each of the problem situations involve measurement division. The child knows the total group of objects to be considered, and must find the number of subgroups. There were 15 cars in Cargo Groups and 20 toy animals in the Animal Group. The cars could be taken across 3 at a time so 5 trips had to be made to get all the cars across. In symbols this could be written:

$$3\overline{)15}\ 5$$

Note that the number of cars to begin with would have to be known for the symbolic form but not if the child chose to "act out" the problem by actually placing cars on the ferry and moving them across. Similarly with the Animal Group the problem was to group the 20 animals in fives and put them with cages. The symbol form for this measurement problem would, of course, be:

$$5\overline{)20}\ 4$$

The second of the two tasks for each of the two problem situations was partitive division. In partitive division the group to be considered is recognized and the number of equal subgroups is given. The task is to find how many would be in each of the equal subgroups. The task in the Cargo Group problem required that the child place the 15 cars so that there are an equal number of cars at each of 3 houses. A systematic partitioning would be to put one by the first house, one by the second and one by the third and then to repeat the process until all the cars were used up. The process would guarantee equivalence in each set (5 cars by each house). The Animal Group task required the child to find how many animals would be in each cage if there were three cages with the same number of animals in each cage and 18 animals to begin with. A partitioning would require the child to put one animal in the first cage, one in the second and one in the third and then to repeat the process until all the animals were used up.

Symbolically the Cargo Group partitioning task could be written:

$$3\overline{)15}\ 5$$

The 3 refers to the number of cars used up per round of the partitioning and the 5 refers to the number of rounds (or the number of cars at each house when the partitioning is complete.)

In symbols the Animal Group partitioning might be written:

$$3\overline{)18}\quad 6$$

The 3 refers to the number of animals used up in one round of the partitioning and the 6 refers to the number of animals in each of the 3 cages after partitioning.

iv. Discussion of Results

In the cross sectional sampling, 10 children at each age level from 3 years to 8 years completed the Cargo Group measurement division task while 5 children at each of these levels completed the Animal Group measurement divison tasks. If all children had been available for the longitudinal sampling there would have been 5 children doing the Cargo Group measurement divison task and 10 the Animal Group task. There was a full complement of children remaining who were 7 years old for the longitudinal task but there was attrition at all other levels with only 5 four-year-olds and 5 five-year-olds available from the original 10 three and 10 four-year-olds.

For partitive division there were 5 children at each age level in the cross sectional sampling who did the Animal Group tasks. There were to have been 10 children at each level in the longitudinal sample for this task but again only the seven-year-olds were fully represented.

Each of the four tasks in each of the samplings for each child was selected from the tapes and analyzed separately, a task at a time. Complete task analyses are presented by Bourgeois (1976) and should be consulted by the reader who requires more detailed information. Summaries and discussion of the analysis are presented here.

1. The measurement division task associated with the Cargo groups apparatus was more difficult for these children than the measurement division task associated with Animal Groups.

Apparatus for the two situations were constructed with the view that equivalent measurement division problems could be based upon them. However, differences in the physical structure resulted in different behaviors at least up to age seven.

Most of the children across the ages four to seven were able to form the required groups for both tasks. Yet a larger proportion of children in the range correctly solved the animal groups measurement division problems even when the order of tasks was reversed. (See sampling procedure.)

The information given to children in both tasks followed the same sequence. First all the objects (cars or animals) were involved in a move from one place to another. Children were then told that a given number of these objects were to be grouped together. Then they were to find the number of resulting groups.

In the case of the Cargo Groups children needed to move the cars on the ferry 3 at a time and unload them on the parking lot opposite and repeat the procedure until all the cars were across. Most children could do this but in the process lost track of the number of trips. Unless the integrity of the groups had been maintained some way in the task they had no record of the number of trips taken. In the Animal Groups problem they merely had to count the number of cages. The Animal Groups apparatus and the nature of the associated problem guaranteed that the integrity of the groups would be maintained.

In applying the criteria for "good" problems it was assumed that the two problems were equivalent and for some 7 year olds and most 8 and 9 year olds they were. But for the younger children the different structure of the two problems affected their ability to do them.

Whether in "acting out" division children tend to reflect more on the consequence of their actions in the case where the integrity of groups is guaranteed or where there is no guarantee of such integtrity is an open question. There is nothing in this investigation which would shed light on it. It does seem reasonable to assume, however, that children have to learn to adopt some means of keeping count of the equal subgroups in measurement division and need some striking examples of what happens when they do not? The Cargo Groups situation may serve this function better and thus be a more effective example than the Animal Groups situation even though children are more successful with the animal groups type of measurement division. Further research will undoubtedly clarify the role of the two measurement division situations in teaching and learning the process.

2. The partitive division task in the Animal Groups situation was more difficult than the measurement division task.

The opposite was true in the two Cargo Groups tasks but the complications in the measurement division task which was discussed in the previous section may have been the contributing factor. In addition, cars were first associated with a ferry and then with houses. In both the Animal Groups tasks children put animals in cages.

The similarities in the Animal Groups tasks would suggest these tasks provide a better basis for comparing the relative difficulty of measurement and partitive division than do the tasks associated with the Cargo Groups. Difficulty differences were more striking in the longitudinal than in the cross sectional sampling.

3. Children responded to a variety of distractions inherent in the situations and associated tasks.

It has already been pointed out that in these problems it was possible for children to give a symbolic response or in other ways reduce the manipulations in solving the problems. However, few children did so. In fact, the striking thing was the elaboration and detail of the manipulations carried out in arriving at a solution. It was evident that children were responding to all aspects of the situation and made little attempt to look for and separate out those aspects essential for a solution. Boys had a tendency to act out the movements of the ferry and the loading and unloading of cars from it in minute detail. Boys tended to be more concerned than girls about the inappropriateness of putting the lion in a cage with other animals while girls showed more concern about the welfare of smaller animals.

Designing interesting problems for children always leads to a dilemma. In making them interesting we include irrelevant details which may distract from the information that is central to the problem. One could hypothesize that all learning involves cutting through some distractions or noise or irrelevancies. The question is how much noise and at what time at any particular learning stage. Dienes (1963) warns against generating so much noise that the child is unable to sort out the relevent abstractions. Skemp (1971) recommends careful control of noise with greater and greater amounts as learning proceeds and deepens. The trick is to introduce enough noise to maintain the interest and motivation of children but to take measures to ensure at all stages that they can cut through it and find the relevant details.

This investigation was not set up to determine which disractions would prevent a particular child from arriving at a solution to a problem and those which would have no effect. However, there were some factors which obviously did adversely affect performance. Some younger children wanted to put all the cars of one color on the ferry in the Cargo Groups situation and did not seem to be concerned that

there were to be three cars on the ferry each trip regardless of color. These children seemed to think that if a problem contained a set of different colored objects then the problem would require them to classify according to color.

In the Animal Groups situation great care was taken by some children to get the lion into a cage with more robust animals such as camels rather than with the mouse. Most children persisted in getting the animals to stand in the cages and as far as possible to group animals of one kind together. The classifying process even when completely irrelevant to the problem was a strong distractor.

Some children wanted to match the color of the cars with the color of the houses in the Cargo Groups situation and others were reluctant to put more than two or three cars by a house because not more than three cars "belong" at one house. Others would not park near the house because the cars could not be driven on the "grass" and the houses were surrounded by green to imitate grass. Boys tended to make car and ferry noises and to make complicated but realistic maneuvers as they drove the ferry across the river and the cars on and off the ferry. Some wanted to put four cars on the ferry at a time rather than three because it would "hold four". Girls were not inclined to simulate reality to such an extent.

There needs to be more attention given by researchers to control of distracting elements in problems such as these. (For some systematically collected information on distractions see Bana (1977), Bana and Nelson (1977), and Bana and Nelson (1978).)

4. Systematic partitioning was carried out by only a small proportion of children.

Only about one child in eight or nine partitioned in a completely systematic fashion. Some distributed objects one at a time but would get mixed up in the order. Others would distribute several at a time. Still others would put a few objects in each group and then try to 'even them out'. Others would first put 3 cars at each house or 5 animals in each cage. They would then proceed to distribute the balance in a more or less methodical fashion. This suggests that requirements of the previous problem (3 cars per ferry load, 5 animals per cage) was influencing the partitive division task.

To guarantee that the requirements of partitioning are fulfilled children should be taught to put an object in one group then in the next and so forth until there is an object in each group. They should then start again in the same order until all the objects are used up. Evidently children are taught partitioning in connection with sharing objects with others and don't completely generalize the process or reflect on what the process means.

We attempted, quite unrelated to the study reported here, to teach a bright four year old girl how to partition. The task was to place a number of blocks into four boxes so all the boxes would have the same number of blocks. She was instructed how to proceed and when she had one round completed put a finger on each block. When the next round was completed she counted the two blocks in each box. Then she said how many there were. This counting went on for several such problems. Then in the middle of doing one of the problems, having counted the objects after a round she exclaimed "Hey, they'll always be the same!" From then on she would do a complete partition, then count the objects in only one group and give her result. That experience and her reflection on it had convinced her that the process would take care of the requirement that there be the same number of objects in each group.

This anecdote is recorded here to lend some support to the belief that partitioning could be taught to children so that it becomes a completely systematic process.

5. Children's spontaneous verbal behavior was varied and seemed to serve different functions at different age levels.

Some children attempted all problems and said not a word. Others were quite verbal. Younger children, as often as not, when they did verbalize monitored their actions (manipulations) as if telling what they were doing and what they were about to do. Older children's verbalizations were more often questions about the apparatus or questions asking for clarification of what the problem task was about.

The whole area of verbalizations and their role in problem solving needs further study. The investigation reported here would have profited by having an expert in language on the research team.

v. Some Concluding Statements

1. Physical differences in apparatus upon which identical problems are based may profoundly affect the child's ability to solve those problems.

2. The partitive division process is more difficult for children than measurement division.

3. Measurement division situations which guarantee separation of subgroups so that these subgroups can be easily counted result in more correct solutions than measurement division situations which make no such guarantee.

4. Requirements of one task on a particular apparatus can influence the results

of another task if it is based on the same apparatus.

5. Mathematically irrelevant aspects of a problem distract children in different ways.

6. Children age six or seven do not learn to partition objects in a completely systematic way.

7. There is a reluctance for children to give a response in purely symbolic forms when a manipulative form is available to them.

B. SPACE AND GEOMETRY

Two problem situations involving representations in two and three dimensions came to be known as Fold-Out Shapes and Projected Shapes. The associated problem tasks were meant to be equivalent.

i. Fold-Out Shapes

Illustration 3

Six squares, four equilateral triangles, and twelve regular pentagons each of side 10 cm were made of 3.2 mm Plexiglas. Alternate edges were fitted with the rough and smooth side of Velcro. The six squares could form a cube, the four equilateral triangles a tetrahedron, and the pentagons a dodecahedron. (Construction details are given on pages 14 to 17 of this report.)

Children were given the following problem tasks:

1. *Cube:* A completed cube was placed in front of the children who were encouraged to handle it and give it a name. The children were then asked to unfold the cube so it would lie flat on the table. They were then asked to fold it into a cube again. Then the following layouts (Figure 1) were presented to the children in turn who were asked if the layouts could be folded to make a cube. (The word most children used was "box". The interviewer used whatever word the child used.)

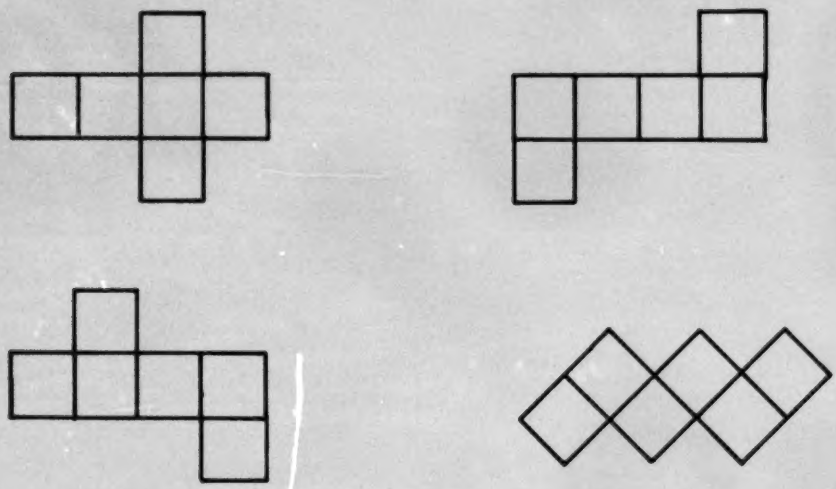


Figure 1

Once children guessed about the foldability they were asked to make the fold or to fold it up into a "box".

When these tasks were completed they were presented with the following in turn (Figure 2):

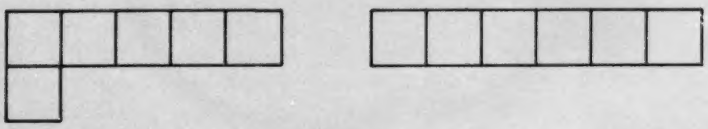


Figure 2

and asked first to predict whether they could make a cube. Once an answer was obtained the child was asked to do it. During all this time a wire cube shape was on the table within reach of the child.

2. *Tetrahedron:* The separate Plexiglass equilateral triangles were placed on the table in front of each child. They were asked to put them together to make a box. A wire-framed tetrahedron of the same dimensions was on the table. Assistance was given to the children who needed it.

   Triangles were then laid out in these arrangements (Figure 3) and the children were asked if boxes could be made of them without detaching any pieces.

Figure 3

After making a prediction children were asked to do it. No assistance was given.

3. *Dodecahedron:* A completed dodecahedron was placed in front of children who were asked to hold it, to dismantle, it and then to reassemble it. No assistance was given other than holding pieces in place and no frame model was supplied.

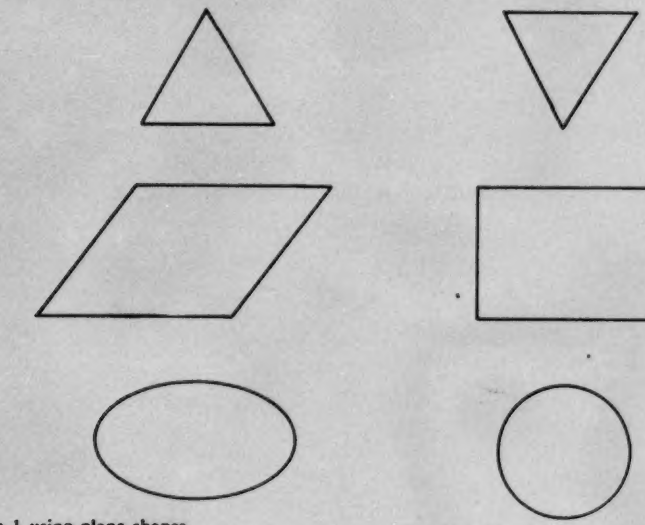
ii. Projected Shapes

Illustration 4

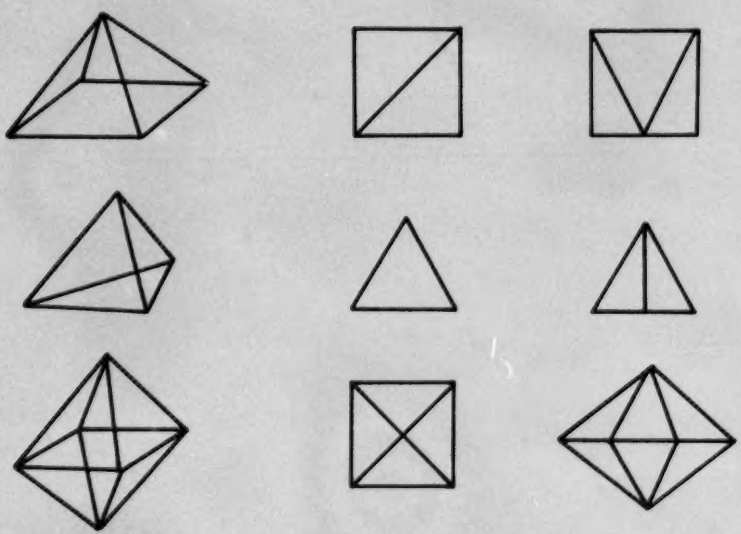
A 35 mm slide projector was used to project a beam into a gray screen.

Three plane wire models, a circle of 10 cm diameter, a rectangle 30 cm x 10 cm and an equilateral triangle 10 cm on a side and a set of three wire models of 3 dimensional shapes - a square pyramid, a tetrahedron, and an octohedron - all with sides 10 cm were presented to children.

They were asked to match the shadows shown in Figure 4.



Group 1 using plane shapes



Group 2 using 3-d shapes

Figure 4

Complete protocols for all the problem tasks in this section are provided by Little (1976).

iii. Mathematical Significance

Although these problems were meant to be equivalent there are some important differences which should be noted. Both of them deal, for the most part, with three dimensional objects and their representation in two dimensions. However, while the projections always were representations in a plane, the fold-outs required the child to combine plane elements into some three dimensional form (cube, tetrahedron, dodecahedron).

The usual way of studying how children represent three dimensional objects in a plane is to show them such objects and ask them to make drawings of the objects. This requires a certain amount of dexterity with a pencil which many of the subjects of this study did not have. The main requirement in the Fold-Out Shapes was that children be able to match the planar elements edge to edge and vertex to vertex. The projections allow children to produce infinitely many representations of an object in a short time and if they have some control, to choose positions of the projected objects that come closer and closer to the needed representation.

A more complete discussion of the mathematical significance of spatial problems such as these is provided by Little (1976).

iv. Discussion of Results

Fold-out Shapes

Cube: Many three year olds can readily construct a cube from a complicated looking two dimensional layout. Nine out of ten readily fit the elements accurately edge-to-edge and corner-to-corner. There was some misinterpretation of the task. Two three year olds and a six year old laid the elements one on top of the other into a shape one of them called a "wallet". Some younger children would fold up one side and not know how to keep it up while they folded another side so were given some assistance by the interviewer. The need for this or any other form of assistance decreased greatly with age although there was at least one child in every age group who needed assistance in constructing the cube.

The reader will recall that the children were presented first with a series of 4 foldable layouts and then with 2 unfoldable layouts. As has been pointed out before, the success rate of predictions for those who would predict was high. This was especially so for the first 4 foldable layouts. However, only one child younger than 7 correctly predicted the unfoldability of the first of the 2 unfoldable layouts. For the second unfoldable layout the rate of correct predictions rose from 30%-40% for children up to 6 and was correctly predicted by far the majority of older children. In fact, all of the eight-year-olds who made a prediction made a correct one. No doubt the prediction that a layout would be foldable was influenced by the four in a row that were foldable. Such habituation would then account for the poor

predictive performance for the first of the two unfoldable ones. It should be noted that predictive efficiency rose immediately from the first to the second unfoldable layout.

Most children displayed some systematic procedure for handling the rather intricate series of folds, turns and fits involved in the construction. For example, in the layout shown below (Figure 5) which some children called an "aeroplane"

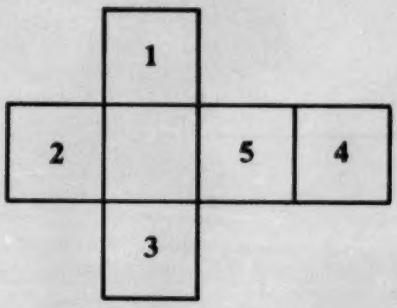


Figure 5

the most frequent procedure was as indicated by numerals. The numeral 1 indicates that that side was folded first, that showing the numeral 2 was folded next, and so on. Younger children tended to fold both ends at once in a global, hugging motion. Of the 60 children who did this problem, only 4 produced an unstable structure in the form of a tunnel.

Before children were permitted to fold from the two-dimensional layout they were asked to predict whether the layout could be folded into a cube ("box" was the term generally used) without moving any of the elements around. Nearly 50% of these children refused to predict. Although the percentage decreased somewhat with age only six of the 10 eight year olds made predictions without resistance. One would think that this reluctance would disappear after children had several layouts to deal with but there was no substantial reduction in the phenomenon in the later tasks. It should be pointed out that when children would make a prediction they were usually correct and the proportion of correct responses increased slightly with age.

Whether this reluctance to predict is a fear of being wrong or an eagerness to get on with the task or a failure to see the need to predict when verification is so easy or whether there is some other reason is still an open question. The behavior does seem to be ideosyncratic in the sense that if children refused to predict in one task they almost always refused to do so in all tasks.

Once children found that the last two layouts were unfoldable they were asked to modify the layouts so that they would fold. About half of the children could modify the first to make it foldable and about three quarters could so modify the second.

Tasks with the cube permitted children to display a wide range of constructional and spatial ability. There was no reluctance on the part of any child to proceed with the folding and, indeed in most cases, with success. Not much evidence was mustered to indicate that children thought of what configurations in the layouts led to foldability or unfoldability. To make an unfoldable layout into one that would fold, one child said he could make it like an "aeroplane" because he knew that would fold. No child gave evidence of reflecting on the tasks further than this which would suggest that experience with the cube was in large part physical and not at all what we would call logico-mathematical.

TETRAHEDRON: Folding the four equilateral triangles to form a tetrahedron may seem a relatively easy task (there are fewer edges and corners to match than in the cube and fewer sides) but it was not easy for these children. Twenty-eight of the sixty could make a tetrahedron and four out of five children could make a tetrahedron when its elements were presented as in Figure 6.

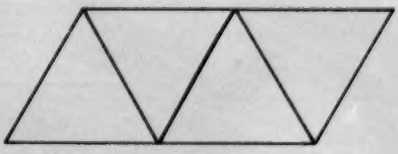


Figure 6

However, when asked to modify the form as in Figure 7,

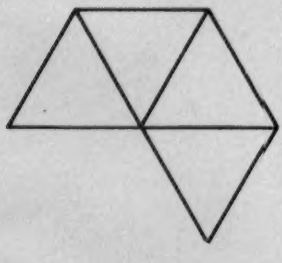


Figure 7

so that it would make a tetrahedron (usually a "box") only one in five could do so. The square based pyramid looked the same to some as the tetrahedron.

When asked to predict the foldability of these layouts the lack of response was dramatic. Three-year-olds and eight-year-olds (cross sectional data) were the only ones apparently willing to make predictions in these circumstances. About two out of three could predict correctly. For the second layout (above) not one five-year-old would predict while the prediction response of four, six and seven year olds was not much better.

The main source of difficulty appeared to be the non-rectilinearity of the fit of the triangles necessary if a tetrahedron is to be made. Children would hold the triangles at right angles to each other and apparently by accident make them come into contact and hence finally solve the problem. There seems to be little in their experience to help them to accommodate to the obliqueness which characterizes the tetrahedron.

DODECAHEDRON: The success with this task was more age dependent than any of the previous tasks. The problem was too difficult for three and four-year-olds though there were solutions at all levels from three to eight. Obliqueness of fit and the larger number of pieces undoubtedly combined to make this a difficult problem for the younger children. All eight-year-olds reconstructed the dodecahedron while 4 out of 5 five, six and seven-year-olds produced reconstructions which were adequate.

The main problem was managing all the pieces as the figure took shape. However, there was still surprisingly little trouble in getting edges and vertices to match. This shape also evoked expressions of interest and delight from children and they did not hesitate to handle it, dismantle it, or attempt to reconstruct it. Dismantling took two distinct forms. Some children very carefully separated each element one from another and spread them out on the table while others just as carefully dismantled the figure so that no piece was separated from the others. Those who ended up with a connected net had some trouble keeping things connected while they attempted to make the reconstruction.

Projected Shapes:

It will be recalled that there were models of 10 shadows that had to be made by projecting images of wire shapes on a screen. Children were first instructed in how shadows of the wire models could be made but younger ones still had difficulty in managing to get a shadow. They would hold the model against their body, obscure the image with their hands or fail to get the image onto the screen through misalignment. However, once the problem of projection was mastered children were good at choosing the correct wire model to produce a given shadow and at getting a fair replication of the given shadow onto the screen.

The easiest problem task was to make a shadow of an equilateral triangle (3). In fact, all subjects but one three-year-old chose the triangular wire model to make the shadow. The correct model for projecting the more difficult shadows (8,9 and 10) was chosen first by 85% of all subjects. The greatest success in this task was among the five to eight-year-olds. Task 10, which was considered the most difficult, was solved by about the same proportion of children from age 4 to 8. Three-year-olds particularly had difficulty adjusting the position of the model to get the required shadow.

The frequency of verbal responses to these tasks was far less than on the Fold-Out Shapes tasks. Making the intricate and sometime confusing moves to get a shadow appeared to take the child's complete concentration. These tasks required more concentrated attention than some of the other tasks and it was impressive to observe how completely involved these small children could get.

The tasks in this problem were appropriate for children in this sample and could form the basis of a series of projection experiences which would prove interesting and profitable for very young children. Once they get the hang of making shadows, they seem to learn quickly how to co-ordinate moves to make a desired one. Many of the drawings of various shapes in their books are merely projections of their edges and vertices in a plane. With projected shadows from the appropriate wire models children can soon learn why Figure 8 is used in books to represent a box.

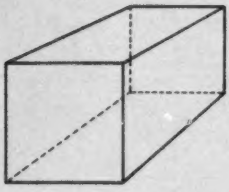


Figure 8

v. Some Concluding Statements

1. Most children in the age range were good at constructing (particularly the cube) but many resisted predicting what form their structures would take.

2. Non-cube tasks presented perceptual and conceptual problems to many children but they were willing workers and often completed the tasks through trial and error.

3. Children were willing to spend a good deal of time erecting structures but there was little evidence to indicate they reflected a great deal about their strategies or their results.

4. Children can make projections from objects to obtain a particular shape but rely on trial and error to get the match. There is little evidence that they reflected on how moves of the object affected changes in its image.

5. Projection tasks were attractive to children and should help them to understand more clearly the derivation of two-dimensional representations of three dimensional objects.

C. REFLECTIONS IN A PLANE

i. Object Reflection

Illustration 5

The apparatus consisted of a board 105 cm x 75 cm surrounded by a wall 5 cm high. Walls were lined with a layer of rubber sheeting to improve reflection of a steel ball (1.5 cm diameter) bounced off it. A spring loaded shooter used to launch the steel ball could be located and rotated (through 360°) in one corner of the board by fitting the pivot into one of 10 drilled holes in the board. A plastic bear was used as a target. As a warm-up the children were asked to tip the bear over by aiming straight at the bear. Then they were asked to tip the bear by bouncing the steel ball off one wall. Finally they were asked to tip the bear by bouncing the ball off two walls. For the bounce shots the bear was placed behind blocks to prevent a direct aim.

ii. Mirror Reflection

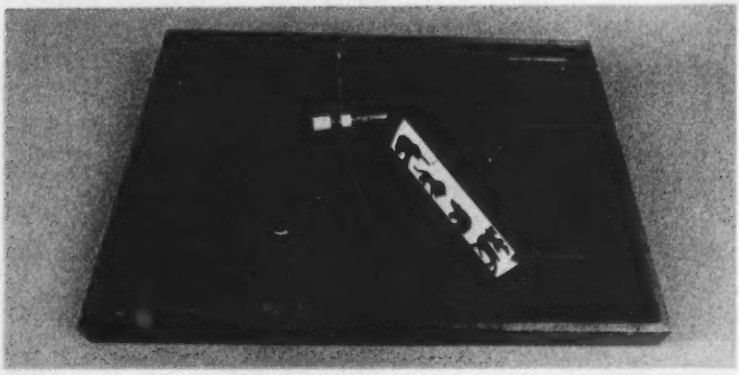


Illustration 6

This was a board identical in size and shape to the Object Reflection board except that the walls of this board were lined with glass mirrors instead of rubber. The shooter in this case was a flashlight that would deliver a fine beam of light mounted on a base and which could move back and forth in a slot cut into the board. The target consisted of an upright strip of cardboard which could be moved about and on which were representations of an elephant, a lion, a bear and a deer. The children were asked to shine the light on animals successively and in a given order, first by shining the light directly onto the animal shapes, then by reflecting off one of the mirrors and finally by reflecting off mirrors on two walls. The room was darkened enough for the light to be plainly visible as the source was moved about.

Detailed protocols for the tasks associated with both mirror and object reflection described briefly here, are provided by Little (1976).

iii. Mathematical Significance

The geometry involved in these situations is similar to that involved in the game of pool. The object (or light beam) is bounced against a wall at a certain angle. If this knowledge is available to the shooter the path of reflections from the walls can be predicted with considerable precision. It was not expected that children in this study would have prior knowledge of this principle but observational opportunities included the possibility that children would, in the course of their experience with the ball and the light, gain some intuitive knowledge of the incidence-reflection phenomenon. The shooters, in each case, could be rotated or moved about (in a restricted way) and this permitted the target to be fixed in one position for each task. An alternative, of course, would have been to get children to place a moveable target so it could be struck by the ball or the beam of light whose path was determined in advance. It should be noted that the light beam could be seen across the face of the board so that effects of movement of the flashlight could be constantly monitored. Feedback on the other apparatus was not so easy since the shooter had to be loaded and fired in order to see where the ball would go.

iv. Discussion of Results

OBJECT REFLECTION

In each of the 4 tasks (two with one reflection, two with two reflections) children were allowed up to 5 shots to tip the bear. A warm-up task involved a

direct shot at the bear. Scoring of responses (+,-) indicated whether the movement of the shooter improved the aim or not. Most of the four-year-olds gave evidence of knowing how to aim directly at an object. They would lean over and sight along the shooter and could hit the target. This behaviour was displayed by an increasing number of older children. Three-year-olds seemed to be more concerned with getting the marble launched than worrying about where it went once it was launched. One three-year-old would launch the ball and yell, "He shoots! He scores!"

In the tasks involving reflections children were asked first which way the shooter would have to be moved to get closer to the object. It will be noted that the shooter would have to be moved the opposite way when there were two reflections than when there was one to effect the same change in the path of the ball. One 4-year-old, two 7-year-olds and one 8-year-old always predicted correctly. All the other children in the cross-sectional sample made at least one error in aligning the shooter and most of them made two or more. The 7 and 8 year olds definitely did better on this task than the younger children.

The next observation was whether the subjects could actually tip the bear. No child was able to tip the bear in all four tasks but three boys (two age 7, one age 8) tipped the target in three of the tasks. By far the majority hit the bear not at all or only in one situation. Correct aiming behavior was used by most 7 and 8 year olds and manifested itself hardly at all in the 3 and 4 year olds. This behavior was exhibited most often by boys. (Only 6 of the 29 subjects judged to be using correct aiming behavior were girls).

About 1 in 3 subjects shot the ball vertically against a wall thus showing little comprehension of the tasks. Most 3 year olds shot directly toward the bear even when the blocks were in the way. They did not seem to be able to comprehend that a target could be hit unless it was aimed at directly.

MIRROR REFLECTION

There were two tasks in this problem. In the first, children directed the beam on the deer and were asked then to move the flashlight so the beam would fall on the elephant. The beam was then directed at the lion, the light switched off and subjects asked to move the flashlight so the beam would be on the bear after the flashlight was turned on and after the beam was reflected once. The second was similar to the first in nature except that there had to be two reflections before the beam fell on the designated figure.

Younger children had difficulty understanding the need for reflections and would, as often as not, aim the flashlight directly at a designated animal and ignore the fact that a reflection (or two) was required in the task. There was not

much difference in success across the age range 4 to 8 for either task but the two-reflection task was definitely more difficult than the one-reflection task. When one considers the amount of informaton provided in the mirror reflection (the path of the light was in most cases plainly visible) it was clear that children across the age range had little comprehension of what was going on. In general they were not able to respond to the information provided.

v. Concluding Statements

1. There is no evidence to indicate that children in this age range were aware of the equivalence of the angles of incidence and reflection in the tasks involving this phenomenon.

2. Children's ability to aim properly improved their performance somewhat in the tasks involving one and two reflections.

3. Children tended to adjust the steel ball launcher by trial and error and did not stop to reflect on how changes in the launcher affected the path of the ball. The same observation applies to the beam of light. In the latter, there wa continuous information fed back but this seemed to be of little use in improving performance.

4. The reflection tasks were too difficult for most children but were interesting and attractive for them to do. All children were eager to get involved in the tasks.

5. The requirement to reflect a ball or beam of light in order to strike a target did not seem a plausible one for many children. Younger ones, as often as not, simply ignored the requirement and attempted a direct shot.

D. CO-ORDINATES

The next pair of problems involved using co-ordinates to locate objects in two and three dimensional space. These problem situations became known as the Parking Lot Grid and the Theatre Grid.

i. Parking Lot Grid

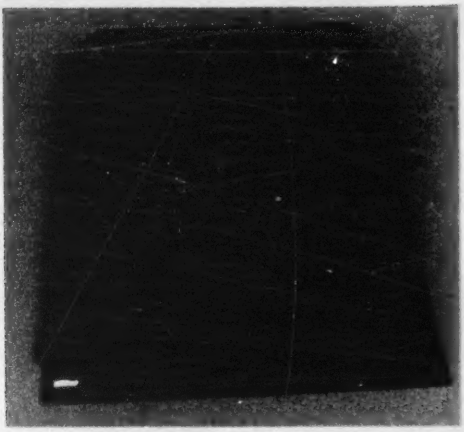


Illustration 7

The apparatus consisted of a masonite base 76 cm x 106 cm x 6 mm painted black. Eight rows of parking spaces each containing 15 bays were constructed from 3 mm x 3 mm balsa strips glued to the base. Each bay was 5 cm x 6 cm. An arrow indicated the entrance to the parking lot.

Two removable masonite strips were fixed along the base to provide co-ordinates for the bays. There were 12 plastic toy cars of assorted colors and 9 laminated parking slips labelled to indicate parking spots.

Children were first familiarized with the co-ordinate system working first with one dimension then the other and finally co-ordinating the two. After this preliminary exercise children were given a parking slip and a car to park. They were to park three cars in this way and were finally shown a parking slip and asked to pick out the car the slip referred to.

ii. Theatre Grid

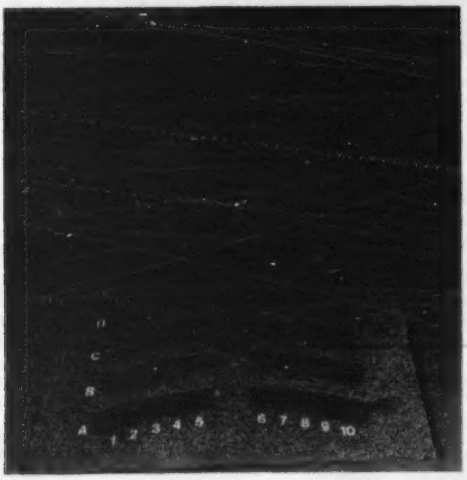


Illustration 8

The apparatus consisted of three levels of seats constructed of masonite and balsa wood. Each level had two blocks of seats 10 cm from each side with a centre aisle 7.5 cm wide. There were five seats in each row both sides of the aisle. A row of seats in each block was 23 cm long and about 4 cm wide slightly curved toward the front. Individual seats in each row were marked out with 3 mm x 3 mm balsa strips glued to the various levels.

Letters were used to label the rows and numerals used to label the seats in the row. The rows of seats were painted black and labelling was done in white. The color of the levels were as follows: yellow for the ground floor, red for the second level and blue for the third level. The rows of the yellow level were labelled A to K, the red level A to F and the blue or top level A to C. In front of row A on each level white numerals 1 to 10 identified the seat in the row. Six dowels 9 mm in diameter and 5 cm long separated the first and second level while 4 dowels of the same dimensions separated levels 2 and 3.

Ten toy people figures and a set of tickets completed the apparatus. The tickets were made of yellow, red or blue ticket board and each had a numeral and letter printed on them. The color of the ticket indicated the level, the letter the row and the numeral the seat in the row.

Children were familiarized with the system of locating seats. They were then given a toy figure along with a ticket and asked to find the seat indicated by the ticket. They were to place their toy figure in the space which represented a seat. After locating four figures in this manner the children were handed a ticket and asked to identify the figure the ticket belonged to.

A complete protocol for these situations is given by Fenrich (1978).

iii. Mathematical Significance

Learning to use co-ordinates to locate positions in space is an important aim of elementary school mathematics instruction. There is a question about how early the notion can be introduced and there has been no exhaustive study of ways it can be introduced to simplify its acquisition. The two problem situations here are both familiar ones for children so may add the interest and incentive for very young children to become involved in using systematic procedures for locating positions in space.

The two-dimensional parking lot requires that children learn to co-ordinate an ordered pair of numbers and to distinguish between (4,3) and (3,4) for example. While the theatre grid requires children to operate in a third dimension it does not

require them to co-ordinate dimensions since each dimension is represented in a different mode. (Colors for levels, letters for rows, numerals for seats in rows). In creating the problem situations it was not expected that the need to co-ordinate in the first situation and not in the other would be offset by the use of two dimensions in the first situation and three in the other. But it was expected that it would make the difficulty level of tasks in the two situations more comparable. Previous clinical work with young children showed that the co-ordinating requirements of three dimensions left them completely lost.

iv. Discussion of Results

PARKING LOT GRID

The first three tasks involved locating cars on the parking lot at (4,3), (2,5) and (8,6). The positioning at (4,3) was used as an instructional task. Responses on the other two tasks were recorded and analyzed.

No three-year-olds were able to park the cars correctly at either (2,5) or (8,6). No four-year-olds were able to locate a car correctly at (2,5) and only one five-year-old could do so. Two four-year-olds and three five-year-olds parked cars correctly at (8,6).

Two distinct patterns of parking emerged. Some children counted out the spaces for the two dimensions while others looked at the scales, apparently matched numerals on the scale to their parking slips, co-ordinated them by finding their intersection and placed the car without counting spaces. For the (2,5) position there were about equal numbers for each of the strategies. For the (8,6) position the older children used the scale matching strategy more than they did the counting strategy. All the 7 and 8 year olds could park the cars correctly with 3 of them at each age level using the counting strategy and 7 of them at each level using the scale matching procedure. No 4, 5 or 6 year olds used the matching procedure to park cars at (8,6) but there were 9 children across this range who employed the counting procedure and parked the cars without assistance from the interviewer. As they gained experience with the tasks more of the older children adopted the scale matching procedure. Most of the children younger than six had trouble completing the task and at best could get one dimension correct.

A final task with the parking lot was to select the parking slip that belonged with one of the parked cars pointed out by the interviewer. Only those children attempted the task who had all three cars correctly parked from the initial parking lot tasks. About half of the children six years and younger failed to select the correct slip for the designated car while all but two of the 7 and 8 year olds could make the proper selection. These were both 8 year old boys. In general, however, children who were able to do all the location tasks without assistance could complete the selection problem.

THEATRE GRID

There were three location tasks in the theatre grid. Toy figures were to be placed (Red,A,5), (Blue,B,8) and (Yellow,E,4). It will be recalled that (Red,A,5) would refer to a seat on the red level row A and seat 5 in that row. Most 5, 6, 7 and 8 year olds were able to locate the figure at (Red,A,5) and all 7 and 8 year olds found the (Yellow,E,4) position. For some reason (Blue,B,8) was found by fewer people than found the other two positions. There was only one incorrect response by the 5 seven-year-olds on the 3 location tasks. Only one three-year-old located a figure correctly and that was with some assistance in the (Blue,B,8) task. The tasks appeared to be too hard for the 3 and 4 year olds but were handled correctly by most 5, 6, 7 and 8 year olds. Although numbers are too small to make a definitive statement it would appear that the task of locating positions in three dimensions when different modes are used to designate each dimension is easier for children in this age range than locating positions in two dimensions when it is necessary to co-ordinate or attach meaning to the components of a number pair. In the cross sectional sampling about 3 out of 5 children located positions in three dimensions correctly and about half the participants located positions correctly in two dimensions.

There was a selection task in the Theatre Grid as well. After the figures were placed correctly in the grid the interviewer asked the subject to find a ticket belonging to a designated figure. Unfortunately the protocol for this task was not followed faithfully in all cases in the cross sectional sampling so interpretation of responses of those who did the task according to protocol cannot be adequately interpreted. However, in the longitudinal data 35 out of 43 children made a correct selection. There was a greater proportion of children who made the correct selection for the theatre than who made one for the parking lot.

In the cross sectional sampling the tickets were made from yellow, blue or red paper to correspond to the color of the levels and the letter and numeral were written on the ticket in black ink. For the longitudinal sample the tickets were made of white ticket board while the letter and numeral were written in yellow. blue or red ink to correspond with the color of the levels. One method of marking tickets did not appear to be superior to the other. Children could respond apparently as well to colored ink as to ticket color.

v. Some Conclusions

1. A strategy of matching and co-ordinating to locate positions in space appeared to grow out of one which involves counting out each dimension where there is little evidence in children's responses to indicate they give much attention to co-ordinating the two.

2. Seven and eight-year-olds tended to look for co-ordinate labels, then find their intersection to get a position rather than to count out each dimension in serial order. Such co-ordinating responses seldom appeared in younger children.

3. Younger children were more successful locating positions in three dimensions than they were in locating positions in two dimensions. The three dimensional tasks provided a different mode for labelling each dimension while the two dimensional tasks provided labels in only one mode (ordered pairs). These younger ones appeared not to have developed the strategy for simultaneous consideration of the referents in the ordered pair.

4. The youngest children to complete the location tasks in both two and three dimensions were 4 year olds.

5. Across the age range (3-9 years) correct response for the two dimensional tasks occured with about the same frequency as for the three dimensional tasks.

E. FACTORS

This pair of problem situations required that children attempt to arrange numbers of blocks in patterns which would reveal the factors associated with those numbers. These situations came to be known as the Factor Platform and the Factor Board.

i. Factor Platform

Illustration 9

The apparatus consisted of a platform made up of 53 cm x 27 cm x 1.3 cm pine with 13 slots 2 cm wide. Slots were separated by strips of pine about 2 cm wide. Across the bottom was a strip of pine to support blocks which could be placed in the slots. The platform was mounted at an angle of about 30 degrees to the vertical to prevent blocks from falling out. The blocks were cubes made of pine and of a size that would fit loosely in the slots. There were 20 such cubes.

The problem tasks associated with this apparatus were to change unequal piles of blocks to equal piles thus finding factors for different numbers by forming sets of equal piles of blocks.

The apparatus was placed in front of children with three blocks in the first slot, one block in the second, six blocks in the third and two blocks in the fourth slot, (3, 1, 6, 2). Children were asked to rearrange the blocks so there would be the same number of blocks in each row (3, 3, 3, 3,). They were told they must only use 4 slots. This was an instructional task and children were helped to complete it if they needed it.

The next configuration was (1, 7, 2, 2,) which was to be rearranged to (3, 3, 3, 3,). Children who could not do this were not required to attempt any further tasks. Those who could were asked in turn to do the following rearrangements:

(3, 3, 3, 3) -- (6, 6)
(6, 6) -- (4, 4, 4,)
(4, 4, 4) into 5 slots.

They were then given 20 blocks and asked to do this rearrangement:

(3, 4, 6, 7) into 5 slots: (4, 4, 4, 4, 4)

One block was then removed from the last slot (4, 4, 4, 4, 3) and the children asked to put the 19 remaining blocks into 4 slots so there would be the same number in each. Finally the configuration (5, 5, 5, 4) was presented and children asked to arrange them in any number of slots so the piles in each slot would be the same.

ii. Factor Board

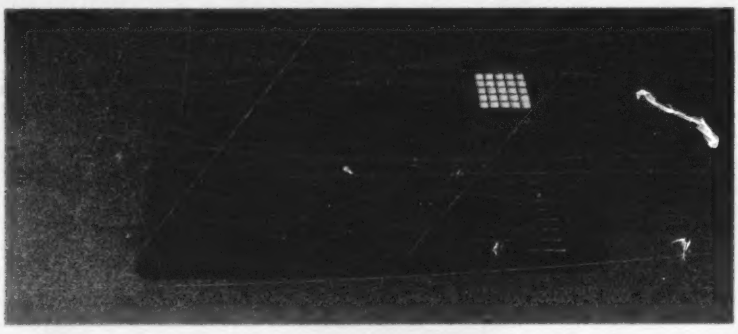


Illustration 10

The apparatus was constructed from 2 cm plywood with a 2.5 cm x 2.5 cm frame. External dimensions were approximately 84 cm x 58 cm. A network of 2 cm squares were formed with narrow unpainted balsa strips glued to the plywood surface. The surface was then painted to form families of colored rectangles as follows:

1. Yellow Family 2 x 2, 2 x 3, 2 x 4, . . ., 2 x 13
2. Blue Family 3 x 3, 3 x 4, 3 x 5, . . ., 3 x 8
3. Red Family 4 x 4, 4 x 5, 4 x 6
4. White Family 5 x 5

All squares remaining were painted black. A set of 25 two cm cubes completed the apparatus.

Sets of factors of a given number may be found by recognizing those rectangles which may be completely covered by a set of new blocks.

The board was placed in front of a child with ten wooden blocks placed in the large black area. Attention was directed to the colored rectangles and squares on the other section of the board and the child was shown how to fit the wooden blocks in the small squares on it.

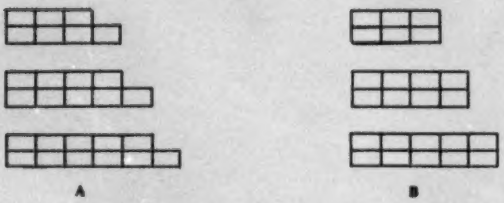
Each child was directed in turn to find the colored space on the board to put the blocks in so that the blocks would just fit the colored spaces. Three tries were given and if children couldn't find the correct space they were shown how to do it.

Two more blocks were then added to the ten original ones and the twelve blocks placed in the large black area. Children were again asked to find a colored space into which all the blocks would just fit. Successful children were then directed to find still another colored space into which to fit all the blocks.

Children not able to fit the twelve blocks were assisted in doing so. Then they were given 2 more blocks and all 14 blocks were placed in the large black area. Again they were asked to find the colored area into which they would all fit. After a maximum of three tries children were assisted in finding the appropriate colored area. In the next problem task the 3 x 5 blue area was pointed out to children. They were allowed to look at it for a time and then, after the colored area had been covered by a piece of cardboard, to select without counting a number of blocks they thought would fit the area.

iii. Mathematical Significance

Young children may see the difference in configurations of blocks in Display A and B but not know that the numbers of blocks in B all have a factor of two, while those in A do not.



Children in this sample were as old as nine years and would no doubt have some instruction in multiplication and division as processes. However, it is not likely that factors and factoring would get much attention in classroom instruction. It was assumed, however, that if means could be found to present a physical model or representation of a number and its factors then any intuitive idea that children may have about the notion might be probed and its development traced.

Factors, factoring and factorization play a significant role in elementary school mathematics. The notion of prime numbers, for example, is derived from learning that this set of numbers all have two and only two distinct factors. Building up sets of equivalent fractions depends in large measure on children being able to handle common factors of two numbers (denominator and numerator). Simplifying computational procedures, tests of divisibility and the fundamental theorem of arithmetic are all developed out of a knowledge of number factors.

iv. Discussion of Results

No other publication is available which provides a detailed analyses of the results for these problem situatons. However, many hours have been spent viewing tapes and organizing analyses in preparation for reporting. Analyses so far indicate that children in the age range involved show little understanding of factors of numbers or even that they exist. There are, however, a few observations that can be made in connection with the analyses and they will be given here.

FACTOR PLATFORM

Task 1 required changing the configuration (3, 1, 6, 2) to (3, 3, 3, 3). No 3 year olds completed this task, about one third of the 4 and 5 year olds could do it and none of the older children failed to do it. A typical response was for the younger children to put all 12 blocks in a pile in one slot. In task 2 which required that the configuration (1, 7, 2, 2,) be changed to (3, 3, 3, 3), there was a definite improvement in responses of 3, 4 and 5 year olds. At these levels, however, some children persisted in piling the blocks in one column. When moving to task 3 in which the (3, 3, 3, 3) situation was to be changed to (6, 6), nearly two out of five children who responded correctly built one column of 6 blocks and then another column of 6 blocks beside it. How they decided how many blocks should go in the first column is not clear from the data but it is almost sure that some children, at

least, thought of partitioning 12 blocks into two equal groups and arrived at 6 in that way. It will be recalled that for task 4 the two columns of 6 from task 3 were to be changed into 3 columns with the same number of blocks in each. No child older than 4 failed to complete the task and about one out of five children who could do the task did so by immediately placing 4 blocks in each of 3 columns.

Task 5 required that the (4, 4, 4) arrangement of task 3 be changed with (2, 2, 2, 2, 2, 2). All children who could do the previous task could do task 5. The strategy of removing all the blocks from the board and then replacing them in the required number of slots seemed by now to have become quite fixed in those children who had used it in previous tasks. Again about two out of five children who could do the task did so by putting 2 blocks in the first slot, 2 in the second and so forth until there were 2 blocks in each of 6 slots.

For task 6 the (2, 2, 2, 2, 2, 2) arrangement of task 5 would be altered so 5 slots would be used with an equal number of blocks in each slot. This is the first non-factor (i.e., 5 is not a factor of 12) and by far the majority of children could give some explanation of why the task could not be done. They used some explanations as "I'd have to take 2 blocks away", "you'd have to have 3 more", and the like. One seven year old said, "you can"t because there is an even number of boxes (slots)' and an 8 year old said, "If you have 12 blocks you just can't put them in 5, you need 6". These last two statements indicate some intuitive awareness that 5 is not a factor of 12. It should be pointed out that not all children knew there were 12 blocks in all or at least considered the information to be redundant if they did know.

In task 7 the number of blocks was increased to 20 and they were arranged to the left of the platform in the arrangement (0, 0, 3, 4, 6, 7). They were to be rearranged into 5 slots with equal numbers of blocks in each slot. There was clearly more difficulty with this task than with previous ones. Whether it was because there were more blocks or because they did not begin in the first slot on the left cannot be determined from the taped sequences of responses. One six-year-old who always removed all the blocks before beginning a task said "First, I have to have one in each slot. That's the easiest way to start." The commonest strategy which got children into difficulty was to put a block in each of the first two empty slots and then try the problem by using 6 slots (See configuration above). A few children put 4 blocks in each slot without hesitating. Again it is not clear how they knew in advance that this would lead easily to a solution.

For task 8 one block was removed from the (4, 4, 4, 4, 4) situation to make it (4, 4, 4, 4, 3) with a total of 19 blocks. Children were directed to fit the blocks into 4 slots so the piles in each would have the same number of blocks. Many children persisted in moving the blocks about to fulfil the requirements of the problem and some of these thought the problem could be done but that they were making some

mistake. Those who saw that it could not be done had a variety of explanations why it could not. Most said you would need an extra block or there would be some blocks left over. An 8 year old said, "I can't do it because you have an odd number" - coming as close as any one to saying that 4 is not a factor of 19.

Task 9 required children to move (5, 5, 5, 4) blocks into any number of slots in equal piles. No child realized immediately that this was not possible. One six-year-old and an eight-year-old made trial piles on the table before saying there was no way of doing the problem. An 8 year old tried columns of 1, 2, 3, 4, 5, 6, in order then said, "I don't think there is anything".

A comparison of the longitudinal and cross sectional data reveals little difference in particular task responses for comparable age levels. The number of children building a single column in response to the tasks decreased in the longitudinal data no doubt because there were no three year olds in the sample while there were in the cross sectional one. Some tightening of protocol in the longitudinal sample appeared to stimulate children to reflect more on the problem tasks. Four and five year olds did not have as much success longitudinally as in cross sectional sampling probably as a result of a less supportive protocol for the longitudinal aspect. In general, however, there was improvement in the quality of response for individual children from 1974 when the cross sectional data were obtained to 1975 when the longitudinal sampling was done.

From the observations outlined above it is clear that few children have even a rudimentary idea of the relationship of a number to its factors. Only a few responded in a manner which would indicate that numbers of blocks made a difference in whether they would fit in a designated number of slots. The responses were in large part purely manipulative and the criterion applied was manipulative. If children reflected on why numbers of blocks would fit in certain configurations and not in others it is not revealed in the responses observed.

FACTOR BOARD:

In the cross sectional sampling, five children at each age level were presented with the tasks associated with the Factor Board. The longitudinal sampling was designed to provide data from 10 children at each level (4-9 years) but the actual number of children still available after a year was as follows:

4 year olds - 6
5 year olds - 7
6 year olds - 10
7 year olds - 10
8 year olds - 10
9 year olds - *8*

Coding of the data for these two samplings has been completed but this coded material has not been thoroughly analysed. Some preliminary analyses have been done and some of the observations will be outlined here. However, any statement made with respect to these preliminary analyses should be viewed as tentative at best.

None of the three year olds completed any of the tasks associated with the Factor Board. The four year olds in the longitudinal sampling did not do much better. However, by age 6 children began to give solutions to all the tasks. There was, however, little evidence of the use of systematic strategies up to age nine. It will be recalled that the yellow areas contained a number of spaces divisible by two and were arranged so that one dimension was 2 (blocks). It was expected that as the number of blocks was increased from 10 to 12 to 14 some children would select only yellow areas as long as the number of blocks they were to place was even. This pattern did not seem to emerge and as often as not, 12 blocks would be fitted to the 3 x 4 area (blue) as to the 2 x 6 area (yellow). The choice of the blue area over the yellow in this case appeared to be a matter of chance rather than of considered strategy.

The last task on the Factor Board was to select a number of blocks which would fit a 3 x 5 (blue) area. The attention of children was directed to the area and then it was covered with a piece of cardboard. It was at this point children were asked to select the correct number of blocks to fit the area and to arrange the blocks in the correct (3 x 5) array. Children as young as 5 would attempt the problem and select out a number of blocks but it was not until children were 7 years old that correct responses came with any regularity. Even among the older children it seemed as likely that a 4 x 4 array would be constructed as would a 3 x 5 array.

v. Some Concluding Statements

1. Children had considerable success in performing tasks associated with factors and factoring in a purely manipulative way. There is little evidence, however, that they attach any mathematical significance to their manipulation. Many eight and nine-year-olds failed to remember the shapes of an array (in this case 3 x 5) when the period of exposure was short and no question was asked in advance to suggest what to look for.

2. There was no evidence that tasks such as the ones provided in the situations involved here have any relevance to a child's subsequent understanding of the relatonship between a number and its factors. Nor did the tasks serve to guide the child to any more than a superficial understanding of factors and

factoring. Children readily built a 3 x 4 array on the Factor Platform or Factor Board but did not seem to relate the two experiences or relate the arrays to the number 12.

3. The willingness of children to engage in the tasks associated with the Factor Platform and Factor Board suggests that these problem situations could form the basis of structured instructional procedures which might, in turn, provide a background of understanding for factors and the process of factoring.

4. Piling of objects in a single column appeared to be an attractive process for some 3, 4, and 5 year olds. The use of this process in connection with the factoring tasks is possibly a manifestation on the part of young children to solve their own problems when they do not have the strategy available or the understanding necessary for solving the problem given. They seem to consider their problem to be a plausible alternative even though they often appear to know they are not addressing the correct problem.

E. SEQUENCES

The pair of problem situations which came to be known as Linear Sequence and Circular Sequence required children to predict the next element in a simple sequence and then to construct sequences of their own.

i. Circular Sequence

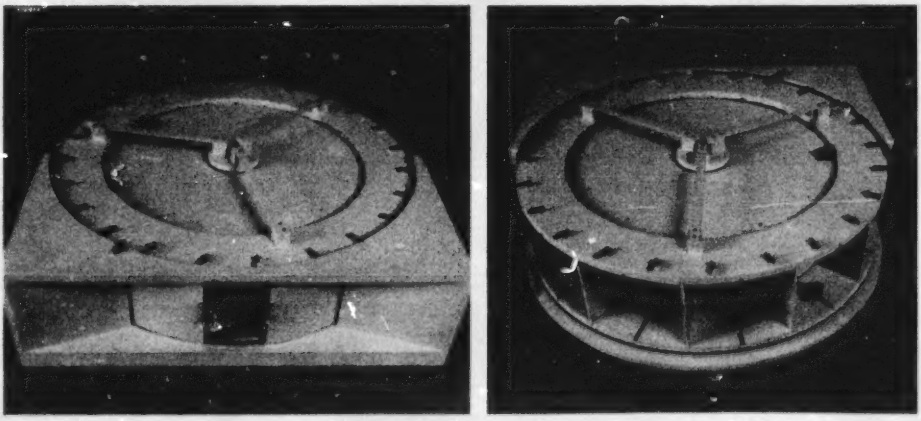


Illustration 11

The apparatus consists of a carousel with a cover which has openings to view the objects placed on the carousel. Here are the details:

1. The carousel of 2 cm plywood has an external diameter of 38 cm and an internal diameter of 28 cm. This is divided into twelve 19 cm x 5 cm compartments by walls of 6 mm masonite. (There is provision for doubling the number of compartments by sliding additional walls into slots in the base and ceiling of the carousel.) The whole thing rotates by means of a vertical cable attached to a separate base platform of 2 cm plywood.

2. The cover, rectangular in shape at the front, is recessed at the back to fit closely to the front half of the carousel. A variable opening is provided by means of doors sliding in slots. The doors are of sheet metal with handles attached for making adjustments. The opening may be adjusted to display one or two compartments at a time. The wall dimensions of the cover are 80 cm x 24 cm x 38 cm. The entire apparatus is finished in white enamel.

3. Another carousel, diameter 27 cm, fitted to the central axle and capable of rotation independently of the outside carousel is of similar material to the rest of the apparatus. Provision is made for cards (12.5 cm x 13.3 cm) to fit into vertical slots on the outside wall of the carousel. Both carousels can be made to rotate together by a locking bolt. Access to the compartments from the rear of the apparatus is easy because it is entirely open.

The apparatus was intended to provide opportunities for children to observe and create sequences.

To begin with each of the 12 compartments contained an object in the following sequence: elephant, red airplane, elephant, red airplane, elephant, white airplane, camel, red airplane, camel, red airplane, camel, white airplane.

Children were seated in front of the apparatus with the sliding doors adjusted so that just one compartment could be viewed at a time. The elephant of the sequence above would be shown first and the apparatus rotated clockwise to reveal what was in the next compartment. Children were asked to identify the object as each compartment appeared. After the third compartment was shown they were asked to guess what they thought would be in each compartment. After a guess was made the carousel was rotated so the guess could be verified. This process was repeated for two complete revolutions. Any child who wanted to could look at the rear of the carousel but was not encouraged specifically to do so.

After two revolutions children were taken to the rear of the apparatus and permitted to turn the platform in any way they chose. They were then asked to

remove all the objects from the compartments and were given 6 blue airplanes, 6 white elephants and 6 little men.

They were then asked to place objects in the compartments from the three classes so that when viewed from the front they would always know what would come up next. Once children completed this task they were taken to the front and the apparatus adjusted so that the piece they placed would appear in the front opening. Their ability to predict was checked by turning the carousel a compartment at a time and, as before asking them to say what they thought would be in the one coming up.

ii. Linear Sequence

Illustration 12

Three rows of 14 boxes or compartments each 6.6 cm x 6.6 cm were constructed from 7.6 cm x 1 cm pine. The overall dimensions of the apparatus were approximately 106 cm x 24 cm x 7.6 cm. Each compartment had a cover of 1 cm pine with a knob so that it could be easily put in place and removed. The covers rested on ledges along the edge of the compartments.

The apparatus was designed to provide children with situations involving patterns which could be color, number or class dependent. With three rows of boxes, three independent sequences could be built up or rows could be combined to provide several variations in sequences.

For this study two rows of compartments were charged with colored blocks as follows: second row - orange, brown, orange, brown, orange, brown, orange, brown,

orange, brown, orange, brown, orange, brown. Third row - orange, orange, brown, blue, orange, orange, brown, blue, orange, orange, brown, blue, orange, orange.

The first row of compartments was not charged with any objects and covers were placed on compartments in all three rows.

Children stood is front of the apparatus while the interviewer pointed out the second row and asked to remove the covers from the first three boxes. (Pointing out the compartment in the row to the extreme left.) They were asked to look at what was in the compartment and then asked to say what they thought would be in the next compartment (the interviewer holding a finger on its knob while children were given time to make a prediction). As soon as children responded they were permitted to see what the compartment held. When the row was completed in this manner children were asked to replace the covers and then remove only those covers with an orange block under them.

The same procedure was used for row three except children were asked to remove only covers which had brown blocks under them.

Children were finally given a set of plastic objects (7 dinosaurs, 7 green cars, 7 toy men) and asked to put the objects in the first row boxes so they would always know what was under each cover when they were all put on.

The covers were then put on and randomly selected covers touched by the interviewer while children said what they thought would be found under them.

iii. Mathematical Significance

Search for patterns or regularity in mathematics should be a purposeful activity for elementary school children. Our dependence on the Hindu-Arabic system of numeration demands that children at a very early age become familiar with certain sequences and sequences within sequences which characterize that system. Odd and even numbers, squares, and other classes important in elementary school mathematics are based on regularly occurring patterns in arrays or factors. The sequence of odd numbers can, for example, be used as a basis for generating the squares of numbers.

Sequences, such as those attributable to Fibonacci, may provide opportunities for young children to see the relationships of mathematics to nature. The whorls in the seed head of the sunflowers and the way branches are attached to stems of some plants are natural manifestations of a Fibonacci sequence. A geometrically constructed representation of such a sequence can be used to illustrate how the chambered nautilus grows.

iv. Discussion of Results

Though the response of children to these sets of apparatus were sometimes complex and highly informative, analyses of those responses have not proceeded far enough to make definitive statements about them. Preliminary examination of the data has, however, yielded enough information to make a few tentative comments.

As with other problem situations children showed no reluctance to get involved in the tasks and, for the most part, the tasks were appropriate for the age range involved. Even some three year olds, for example, showed some understanding of the simple alternating sequence in the first task on the Linear Sequence apparatus. They did, however, have difficulty building their own sequences.

Prediction was an essential part of the first tasks in both the Linear and Circular Sequence situations. The same reluctance to predict as had been observed in other problems where prediction was involved showed up in these problems. The phenomenon was manifested mostly in the younger children, who tried to remove the investigator's hand from the carousel so that they could turn it and see what was in the compartment before they would say what they thought might be in it. In the linear sequence they would try to remove the investigator's hand from the knob of the cover to the compartment whose contents they were asked to identify. Though it is not possible at this stage to report percentages of children who responded to prediction in these ways, it is evident that some children would hesitate to predict even though they knew perfectly well what was in the indicated compartment.

It has already been pointed out that even young children can think in terms of simple alternating sequences when orange and brown blocks were used. However, there does not seem to be much evidence that children before the age of six can build up their own sequences from given sets of objects. A common strategy used by children across the age range was to start out with objects of one kind (say dinosaurs) placing them in compartments in order until they were all used up. Then they would start with another set (say toy men) and use all those objects up. When asked what was in designated compartments, these children were quite successful when the compartment indicated was near the place where one set was discontinued and another started. They usually made at most one error in predicting what they had placed in the compartments. At this point it is not possible to say that children were thinking in terms of minimizing errors in their predictions.

In any case, it seems to be an effective strategy for children who did not fully understand or observe the patterns in sequences they had dealt with in earlier tasks.

In the Circular Sequence children might have predicted that an animal would follow an airplane and would have been correct. However, no child whose responses have been examined did this nor did any of them notice that a change in the color of the airplane signalled a change in the kind of animal which would appear next. This seems surprising when one observes the complexity of some of the sequences created by older children. Some involved as many as 4 elements in a sequence, and there were some with sequences within sequences.

Tasks which required children to fill in a missing element in a sequence were not included in this study. Such tasks may tap an important aspect of children's growing notion of sequence patterns and should be included in future studies in this area.

When young children were doing the tasks associated with the Circular Sequence many of them seemed to be impressed with the large number of objects in the sequence. Although there were only three elephants in the sequence on the carousel, children's comments suggested they thought they had seen many more.

EVALUATION

A. METHODOLOGY

The research referred to throughout this monograph involved the development of good problems for young children, working out protocols for use with the problems, recording the responses of children doing the problems and finally analyzing those problem solving responses. The methodology used in the research is not commonly used by investigators in mathematics education so some evaluation of certain procedures used is needed. It is admitted that internal evaluation of this type may lack objectivity but an attempt has been made to minimize the danger by commenting mainly on those innovations that are unfamiliar to many research workers. (For external evaluations of parts of the research concerned readers are referred to Reys and Bestgen, 1978 and Lindquist, 1979.)

The methodology would best be described as clinical with the video tape recorder serving as the data gathering device. All the responses of all children doing the problems in a laboratory setting were recorded on one half-inch video tape. Their responses formed the basis or provided the raw data for all subsequent analyses. No attempt was made in advance to select any particular responses or eliminate others. The analysts would seek out regularities and form in the data but these, if they existed, would have to come out of the data themselves. In any case, reporting would consist of detailed descriptions of responses made in controlled situations.

Video tape is an effective data-gathering device for this kind of research. Properly used the technology is capable of recording faithfully everything that a child does or says. Barring breakdown in the equipment, there need be no fear of losing information or missing some part of a response. It is all there. Once recorded, the data can be analyzed at leisure simply by running the tape through. Different analysts can select any part of the data that is interesting to them. One analyst may run a tape through a number of times, each time looking at some different or particular aspect of the data. For example, verbal responses might be analyzed for their general function or for a particular function in the problem solving process. Response to extraneous information in a problem may be monitored at one time and problem solving strategies leading to a correct solution at another. A different template, so to speak, can be used through which to view the data whenever the tape is re-run. Data are complete and rich so several re-runs are usually necessary. Whatever the information one seeks, reliability checks can readily be made by having a number of analysts encode the same taped material for the same information and then compare their analyses.

On taped data it is relatively easy to detect flaws in procedure or in use of protocols. Slight modifications in protocol, which might be missed in direct or simultaneous observation, and which can have a profound effect upon the way children respond in a particular problem situation need never be overlooked in taped data.

Despite the obvious advantages, there are some serious shortcomings in using video tape as a receptacle for observations. Its use invariably involves introducing some rather imposing machinery (and its operators) into the situation. Means can be found in the laboratory for obscuring some of the apparatus but it cannot be completely hidden. What effect any of this has on responses of children is simply not known but it could be significant. In the investigations reported here there was little apparent distraction caused by equipment as long as the vidio monitor was kept out of sight. A rather serious drawback for us was that children were obviously responding to color in some of the problem layouts and we had only black and white capability. Many of the responses which were apparently being made to color could not be verified and thus could not figure in the analyses.

Another problem when responses are made on continuous tape and analysis are made in segments (each problem separately for each child) is locating the segments efficiently. Regardless of the care used in documenting location of segments it is always time-consuming to find a required taped segment. Finding a segment of a few metres in a tape hundreds of metres long is tedious and wasteful of time. Time used in searching for data is unproductive and expensive because it is time which otherwise could be devoted to analysis. Development of the video disc with random location should prove to be an effective solution to this problem.

Raw taped data cannot usually be analysed without first encoding the taped information on paper. Such encoding is not an easy task for most people. In fact, it is exceptional to find an analyst capable of spending as much as two hours continuously at an encoding session. Most can tolerate it for periods in the order of half an hour before a substantial break. Cost of encoding and analysis is almost always underestimated and a shortage of funds is typical in studies which involve locating, encoding and analysing substantial amounts of taped material. Measures should be taken by investigators planning to use video tape to keep the amount of taped material as small as possible so that analysis can be thoroughly done. It is my guess that there are, in North America, hours and hours of possibly important taped data lying in storage for want of funds to do a proper analysis.

The ideal situation for taped information is a short observation on one child in one setting which can be analysed independently. A group of such observations on a number of children could constitute a study with each observation an entity which could be combined directly with others of the same kind. I emphasize using a single child because all our attempts at taping responses of groups of children have failed to provide us with any usable data.

Investigators planning to use taped data should not skimp in obtaining and making use of the best available technology. With younger children, for example, color capability is essential. The video disc mentioned previously is a recent development which provides the means of locating, efficiently, needed segments of information embedded in a mass of other information. To ensure the proper operation of equipment there should never be a shortage of trained technicians. Attempts of investigators to operate video equipment and interview children at the same time as often as not results in poor quality picture and sound and frequently in unusable data. The resulting waste of research time on the part of investigators, their subjects and their associates is intolerable.

Many beginning researchers hesitate to develop a clinical study because the method is not always accepted as a productive procedure in mathematics education. Their research advisors usually understand the basis of classical statistical research and apply statistical procedures to data with confidence and skill. Graduate study in mathematic education includes the mastery of probabalistic designs almost as an initiation rite while little attention is given to developing the skills of observation and description which are the basic processes of clinical research.

Recently there have been signs that all is not well in mathematics education and that classical research designs are at least partly responsible for the difficulty. Starting in the early seventies the Center for Science and Mathematics Education with the cooperation of ERIC (Educational Resources Information Center) at the Ohio State University began publishing a quarterly journal called: Investigations in Mathematics Education (IME). In this journal are published abstracts and critiques of studies in mathematics education already completed and reported in other educational journals. Of particular interest is the nature of the criticisms levelled at this research. In 1972, 1973 and 1974 there were over one hundred fifty studies abstracted in IME. The abstracts were accompanied by abstractors' comments. Nearly 80% of these studies were criticized either because of some flaw in design, or because insufficient information was given to make out precisely what procedures had been used, or that conclusions claimed were unjustified by findings, or the problem itself was considered to be trivial (Nelson, 1976). What is significant about the nature of the investigations reported in IME is that nearly every one used what might be called a classical statistical design.

Because researchers fail to apply a design with wisdom and skill is not reason for condemning the design. However if a design fails in its application to research problems some corrective measures need to be taken. One course seems to be the development and use of alternate designs. I would not want to see statistical designs disappear from the repertoire of research workers in mathematics education. But I would want such investigators additionally to learn how to apply clinical methods to mathematics education research and to gain a high level of skill in the process of observation, description and conceptualization. Those clinical

processes hold the key to many important developments in mathematics education and undoubtedly in many other curriculum areas as well.

One could argue that a rich literature of observation, description and conceptualization, which is the very basis of the scientific method, is essentially missing in education. In medicine, biology, astronomy and related disciplines application of rigorous scientific enquiry was preceeded by years of simple observations and description of them. One can read the descriptions of Ptolemy, and Copernicus, and Galileo and see how their observations led them to their conceptualizations of the movement of the planets in relation to the sun. Later applications of more rigorous methods have proved Galileo to be substantially correct in his concept of the solar system although the notion of a heliocentric system was considered heretical at the time and caused him some trouble. A few decades ago the study of biology consisted largely of the morphology of living forms.

A literature of simple observation, description and conceptualization is as conspiciously absent in education as it is apparent in other areas which employ the scientific method. It is as though in the urgency to apply strict scientific principles in educational research there was not time to fill in the gap. At the turn of the century E. L. Thorndike advanced his particular brand of connectionism as a panacea for learning when it had little observational evidence to support it. Observations and descriptions of actual learning behavior were eventually filled in and showed the severe limitations of his bond theory but not before Thorndike's disciples had spread it far and wide. Since then education has been characterized by the adoption of fads that had to be modified when simple observation showed them to be just that. A substantial literature of observation and careful descriptions of those observations would decrease the tendency to faddishness in education and would provide a better basis for more rigorous enquiry.

B. CONTRIBUTION OF THIS RESEARCH TO MATHEMATIC EDUCATION

It has been pointed out before that the methodology used in this research is not a familiar one in mathematics education and that there is not much in educational literature at all that consists of simple observations and descriptions of such observations. Even so, it was apparent from the outset that the objectives of this research could best be met by making use of that kind of methodology. When an unfamiliar procedure is adopted there is always the problem of communicating findings to colleagues in the field. After all, few of the requirements of classical research design were met in the current study. Some of the findings reported here have been subjected to more rigorous procedures (Bana, 1978, Bana & Nelson, 1977) but all statements about the research need further verification.

The following statements appear to be supported by the research:

1. In almost all cases children were eager to do the manipulations associated with the problem situations. These manipulations involved grouping, partitioning, piling, projecting, constructing, factoring, sequencing, locating objects in space, and the like. In fact, children became involved in many mathematical processes in a manipulative way which would not show up in formal mathematics instruction for several years. Children's spontaneity in dealing with the problems suggested that they welcomed being able to impose the required order or form on the objects involved. Whether or not this early involvement in mathematical processes has any bearing on later more formal learning of the processes is not known.

   The fact that even older children produced a purely manipulative solution when they were capable of giving a more sophisticated symbolic solution might suggest that the physical basis of the processes had not been well established before they were required to provide symbolic representations of the processes. It was as if they trusted manipulative solutions more.

2. Children are easily distracted by irrelevant attributes in a situation. When told that the ferry could take three cars on each trip some chidren put four cars on because the ferry "would hold four". Some would put all the red cars on the ferry rather than three as if the problem were to classify the cars according to color rather than to arrange them in groups of three. Animals in cages were all made to stand even though the problem had nothing to do with how the animal was placed. Some children would not do the problem because the cars couldn't be parked "on the grass". Others would match the color of cars with houses or would not complete a task because not more than three cars belonged to one house. Some children would become involved with elements of the apparatus and want to know the make of a toy car or the fate of a mouse in the same cage as a lion.

   The frequency of response to any one of the irrelevancies was not high - but response to some irrelevant aspect or other of a problem situation persisted among children at least up to age five or six. In some cases, attraction to irrelevancies was so strong that children failed to solve a problem, in others they created a problem of their own to solve and in others they first responded to an irrelevancy, then ignored it and finally went on to a correct solution.

   In a follow up study to this one with 360 children in first, second and third grades, doing problems in which irrelevancies were deliberately introduced, 66% were distracted by an irrelevant element, (Bana, 1977). Many children who responded to irrelevant information in the follow-up study seemed to be solving a problem different from the one assigned. In fact

making responses to irrelevancies appeared to be related to their ability to pose a plausible alternative problem to solve.

Gelman (1969) in a study involving children who were unable to conserve number demonstrated that they made conserving responses once they learned that spatial arrangements of objects in the tasks were irrelevant. Gelman taught children to ignore spatial arrangements when they were comparing the number of objects in one set with those of another. The notion that non-conservation is a special case of children responding in a predictable manner to irrelevancies (noise) introduced into a situation is an attractive one. If that is the case then conservation/non-conservation responses explain only a small part of they way children respond to irrelevant attributes in situations. One might suggest that if children respond predictably in one situation (i.e. where conservation is concerned) then responses to noise in general might be predictable. Our findings have supplied evidence that when noise is introduced into a problem situation some younger children, at least, will respond to it. It remains to be shown whether such responses happen with enough regularity to make them predictable or not.

What role distractions play in learning is not at all clear. Since younger children respond to more of them than older children and with greater frequency in problems such as the ones used here there is little doubt that learning and maturity are important factors. It appears that certain types of learning involves cutting through the "noise" of distraction. What is not clear is the amount of noise children can tolerate and how much of it there should be at what stage of learning. Could it be that in particular types of learning activities, at least, noise could be monitored and controlled to optimize that learning? At the present moment the methods used by Bana (1977) appear to offer an effective if not the best approach to the question.

3. A number of tasks required children to predict what some particular outcome would be. About half the children showed a definite reluctance to make such predictions. For example, in the devices which contained sequences children would try hard to get a look at what was in the next compartment rather than say in advance what was in it. It may be that some children who would not say could have made a correct guess. When they were asked whether a two dimensional layout would fold up to make a box many children would proceed to fold up the layout rather than say in advance whether it was foldable or not.

Why there is this resistance to predict an outcome when there is no penalty for making an error and when there is good information upon which to base the prediction is not clear. In most cases a simple manipulation would have revealed the correct answer so some children may have preferred to make

that handy manipulation and be sure of their answer. Another explanation might be that children simply did not know how to predict or it may be that their knowledge of the situation was purely manipulative. If the experience was a manipulative one there would be little, if any, reflection or meaning given to the manipulations. In such cases children would need to manipulate to verify and so be reluctant or unable to make a guess. Another possibility is that the response to being asked to make a prediction is a learned one. It does not appear that it is school related, however, since about the same proportion of pre-school age children and school age children were reluctant to make a prediction when asked to do so. A final possibility is that the phenomenom was an artifact of the particular procedures and materials used in the investigation.

When children made predictions willingly or with minimal urging they almost always predicted correctly. Their willingness to commit themselves may indicate confidence that they could safely make a prediction or it may mark a higher level of processing of the information available. Whatever the case, asking children to predict when there is information upon which to make a prediction seems to stimulate them to reflect upon that information. It would appear that instructional programs in elementary school mathematics should provide opportunities for children to practice making informed guesses about a variety of outcomes.

4. The decision to develop guidelines or criteria for the creation of the problems of this study seemed to be a good one. This is, of course, not the first time such a strategy has been used. Most of the problem tasks were attractive to children and as has been pointed out before they were eager to engage in the tasks. One troublesome observation, however, is that problems when measured against the criteria seem to be equivalent but produce quite different responses on the part of children. Another is that some problems are better than others in the sense that children respond more willingly in one situation than another. In some cases the mathematical ideas underlying the problems were too advanced for children in the sample but in others the physical apparatus and the associated manipulations appeared to be confusing. It was obvious, for example, that few children had any notion of factors and factoring and the manipulations required in the reflection tasks were too complicated for the younger children.

5. There is nothing in the investigation to indicate what significance any of the apparatus or related problem tasks would have in developing curriculum materials and experiences. In other words, teachers should not assume that procedures used here would be effective in teaching related mathematical ideas to children. In fact, no assessment of instructional worth was attempted at all. That is not to say that teachers should hesitate to work out and

evaluate teaching sequences in which the material of this investigation would play some role. But such advance attention would be essential. Manipulative responses may sometimes be more indicative of a child's understanding of a mathematical process or principle than responses generated on a more verbal level. Devising apparatus and accompanying monitoring techniques could add substantially to an evaluation skill and some guidance for developing that skill may be inferred from reports of this investigation.

C. CONCLUDING STATEMENT

An attempt has been made in this monograph to describe in some detail the research procedure used to study problem solving responses of young children. The description includes a discussion of the resulting analyses of those responses. It has been pointed out that while the results need to be enriched by more rigorous methods, clinical studies can be used effectively to indicate new and fresh directions for mathematics education research. Such clinical procedures can also provide a basis for conceptualizing the mechanisms by which children engage in problem solving tasks.

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